13.10.08
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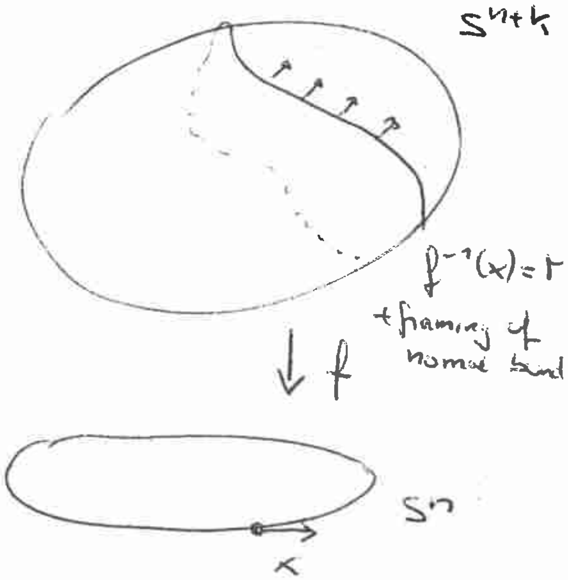
TMF: Overview

Degree of a map: # points in inverse image

1930's Pontryagin:



$$\pi_{n+k} S^n \hat{=} \text{framed } k\text{-manifolds}$$

"Framing" is not very geometric structure.

framing $\searrow$ Stable homotopy groups $\swarrow$

1960's   Spin-Structure (Atiyah-Singer index theory) $\longleftrightarrow$ image of J (K-local homotopy theory (Adams' e-invariant))

Some stable homotopy groups of spheres. $\pi_k^{st} S^0$

$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$
								$\uparrow$ SU(3)	$\uparrow$ U(3)	$\uparrow$ Sp(2)			Sp(1)×Sp

framed manifolds not detected by "geometry" $\rightarrow$

not detected by "geometric" invariants

14	15
$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/480$
G_2	$U(1)\times G_2$

tmf will explain this portion of stable homotopy groups (although not entirely geometric).

Eisenstein series: $\sigma_k(n) = \sum_{d \mid n} d^k$

$E_2 = 1 - 24 \cdot \sum \sigma_2(n) \cdot q^n$ confr: π_3^S

$E_4 = 1 + 240 \cdot \sum \sigma_3(n)\, q^n$ confr: π_7^S

$E_6 = 1 - 504 \cdot \sum \sigma_5(n) \cdot q^n$ conf: π_{11}^S

$E_8 = 1 + 480 \cdot \sum \sigma_7(n) \cdot q^n$ conf: π_{15}^S

come up from Bernoulli numbers

1970's: Quillen, Morava: formal groups $\longleftrightarrow$ complex cobordism

$\Downarrow$ Adams-Novikov SS

$\pi_k^s S^0$

Miller, Ravenel, Wilson: Chromatic filtration

$K(n)$ localized homotopy theory

$n=0 \iff$ degree

$n=1 \longleftrightarrow$ K-local homotopy

$n=2 \longleftrightarrow$ next "new" piece of $\pi_k^s S^0$

Chromatic picture relates automorphisms groups of formal grps to homotopy groups

$\mathrm{Aut}(f_g) \longleftrightarrow$ homotopy groups

p-adic Lie groups

$n=2$: p-adic units $\mathbb{Z}_p^\times$

$n=2$: p-adic quaternion algebra

1980's: Ochanine genus

M oriented $\longmapsto$ level 2 modular form

Witten - related Ochanine genus $\to$ geometry on loop spaces

M spin, $\frac{P_1}{2} = 0 \longmapsto$ level 1 modular form

"Wittengenus"

Landweber - Ravenel - Stong: elliptic cohomology

1990s: Hopkins, Mahowald, Miller

Elliptic curve: $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$