

Analytic theory (of modular forms).

Doubly periodic analytic functions.
w.r.t. lattice $\Lambda \subset \mathbb{C}$,

$$f(z+\lambda) = f(z), \quad \lambda \in \Lambda.$$

If f is also holomorphic, then f is constant. Must have at least 2 poles in a fundamental domain for Λ . First guess

$$\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2}$$

but this does not converge. Correct to

$$P(z, \Lambda) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

which converges absolutely. This does not obviously satisfy $P(z+\lambda, \Lambda) = P(z)$. However,

$$P'(z+\lambda, \Lambda) = P'(z) \quad \left(= -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} \right),$$

so $P(z+\lambda, \Lambda) - P(z) \equiv$ a constant c .

Take $z = -\frac{\lambda}{2}$ to get $\wp(\frac{\lambda}{2}, \Lambda) = \wp(-\frac{\lambda}{2}, \Lambda) + c$

But $\wp(-, \Lambda)$ is even, so $c = 0$. The

field of doubly periodic analytic fcts

is generated by \wp and \wp' , but \wp

and \wp' are not algebraically independent.

Look at Taylor expansion

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{n \geq 1} (n+1) G_{n+2} z^n$$

$$G_k = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^k} \quad \text{Eisenstein series}$$

$$G_k = 0 \quad k \text{ odd.}$$

so starts out

$$\wp(z, \Lambda) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4$$

$$\wp'(z, \Lambda) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3$$

This gives the equation

$$(\wp')^2 = 4\wp^3 + 60G_4\wp + 140G_6$$