

Analytic theory (of modular forms).

Doubly periodic analytic functions.  
w.r.t. lattice  $\Lambda \subset \mathbb{C}$ ,

$$f(z + \lambda) = f(z), \quad \lambda \in \Lambda.$$

If  $f$  is also holomorphic, then  $f$  is constant. Must have at least 2 poles in a fundamental domain for  $\Lambda$ . First guess

$$\sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^2}$$

but this does not converge. Correct to

$$P(z, \Lambda) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right),$$

which converges absolutely. This does not obviously satisfy  $P(z + \lambda, \Lambda) = P(z)$ . However,

$$P'(z + \lambda, \Lambda) = P'(z) \quad \left( = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^3} \right),$$

so  $P(z + \lambda, \Lambda) - P(z) \equiv$  a constant  $c$ .

Take  $z = -\frac{\lambda}{2}$  to get  $P(\frac{\lambda}{2}, \Lambda) = P(-\frac{\lambda}{2}, \Lambda) + c$

But  $P(-, \Lambda)$  is even, so  $c = 0$ . The

field of doubly periodic analytic fcts

is generated by  $P$  and  $P'$ , but  $P$  and  $P'$  are not algebraically independent.

Look at Taylor expansion

$$P(z, \Lambda) = \frac{1}{z^2} + \sum_{n \geq 1} (n+1) G_{n+2} z^n$$

$$G_k = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^k} \quad \text{Eisenstein series}$$

$$G_k = 0 \quad k \text{ odd.}$$

so starts out

$$P(z, \Lambda) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4$$

$$P'(z, \Lambda) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3$$

This gives the equation

$$(P')^2 = 4P^3 + 60G_4 P + 140G_6$$

Emb. of  $\mathbb{C}/\Lambda$  as cubic plane curve  $C$

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{P}^2$$

$$[O : O' : 1]$$

with equation

$$y^2 = 4x^3 + g_4x + g_6$$

Genus  $C = 1$  :

$$\dim H^0(C, \Omega^1) = 1$$

on  $\mathbb{C}/\Lambda$

$$\frac{dx}{y}$$

$$x = x(z)$$

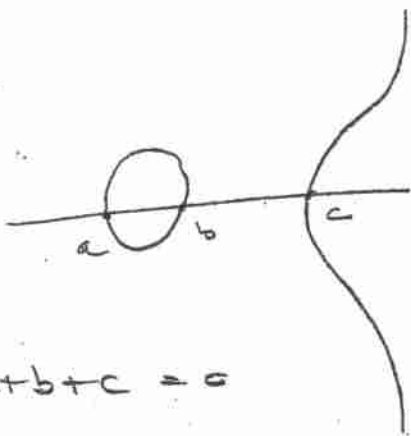
$$y = y(z)$$

$$dz$$

$$\frac{dx}{y} = dz$$

$$y = x'$$

Group structure



point at  $\infty = 0$ .

moduli space of

lattices  $\Lambda \subset \mathbb{C}$

$$\Lambda \sim t\Lambda \quad 0 \neq t \in \mathbb{C}$$

moduli space of

plane cubic curves

$$\sim$$

Want not only to study invariant fets.  
( $f(t\Lambda) = f(\Lambda)$ ) but fets. that satisfy

$$f(t\Lambda) = t^{-k} f(\Lambda)$$

For example,  $G_k(t\Lambda) = t^{-k} G_k(\Lambda)$ . These  
will correspond to sections of tensor  
powers of a line bundle.

$\langle w_1, w_2 \rangle =$  lattice spanned by  $w_1, w_2$

$$\sim \langle 1, z \rangle \quad \text{im}(z) > 0$$

$$\langle cz+d, az+b \rangle \sim \langle 1, \frac{az+b}{cz+d} \rangle, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

so equation  $f(t\Lambda) = t^{-k} f(\Lambda)$  becomes

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Space of lattices

$$= \mathbb{H} / SL_2(\mathbb{Z})$$

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \text{im}(z) > 0 \}$$

Def An analytic modular form of wt  $k$  is a holomorphic fct.  $f$  on  $\mathbb{H}$  s.t.

$$(i) \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

$$(ii) \quad \lim_{t \rightarrow \infty} f(it) < \infty$$

Now consider moduli space of plane cubic curve. Plane cubic

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

over  $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ . Transf.:

$$I \quad \begin{cases} x \mapsto x+r \\ y \mapsto y+sx+t \end{cases}$$

$$II \quad \begin{cases} x \mapsto \lambda^{-2}x \\ y \mapsto \lambda^{-3}y \end{cases}$$

Again do not only want to consider the functions  $(f \in A)$  invariant w.r.t. these transf.

Def An algebraic modular form of wt  $k$  (over  $\mathbb{Z}$ ) is an elem.  $f \in A$  s.t.  $f$  is invariant under (I) and transf. as  $f \mapsto 2^k f$  under (II). "

Make  $A$  into graded ring  $|a_i| = 2i$ , then  $f \mapsto 2^k f$  if and only if  $|f| = 2k$ .

Compute the ring of alg. modular forms following Deligne's paper in Antwerp Proceedings.

First invert 6. Then

$$y \mapsto y - \frac{1}{2}(a_1 x + a_3)$$

transforms the gen. equation to

$$y^2 = x^3 + \frac{b_2}{4} x^2 + \frac{b_4}{2} x + \frac{b_6}{4}, \quad b_i \in A.$$

Also

$$x \mapsto x - \frac{b_2}{12}$$

transf. this eq. to

$$y^2 = x^3 + \frac{c_4}{48}x + \frac{c_6}{864}, \quad c_4, c_6 \in A$$

It turns out that

$$\frac{c_4^3 - c_6^2}{1728} = \Delta$$

and that this is the discriminant of the curve  $C$ . Moreover, the last form of the equation is unique.

So we have produced a map of graded rings

$$M = \mathbb{Z}[c_4, c_6, \Delta] / (c_4^3 - c_6^2 - 1728\Delta)$$

→ invariants in  $A$

which is an is. after inverting 6

Prop (Tate) This map is an iso.

To prove this, it suffices to check this after localizing at 2 and 3. By Nakayama's lemma, it suffices to show monomodulo 2 and 3. Modulo 3, consider

$$M \otimes \mathbb{Z}/3 \rightarrow \mathbb{Z}/3 [b_2, b_4, b_6] \hookrightarrow b_2^{-1} \mathbb{Z}/3 [b_2, b_4, b_6] \\ \rightarrow b_2^{-1} \mathbb{Z}/3 \mathbb{Z} [b_2, b_6] \quad (b_4 \mapsto 0).$$

Calc.

$$c_4 \mapsto b_2^2 - 24b_4 \equiv b_2^2 \pmod{3}$$

$$c_6 \mapsto b_2^3 \pmod{3}$$

$$\Delta \mapsto b_2^3 b_6 \pmod{3}$$

so easy to check this is monom. Modulo 2, invert  $a_1$  and set  $a_3 = a_4 = 0$

$$M \otimes \mathbb{Z}/2 \rightarrow a_1^{-1} \mathbb{Z}/2 [a_1, a_2, a_6]$$

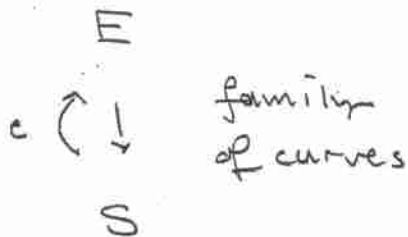
$$c_4 \mapsto a_1^2$$

$$c_6 \mapsto a_1^3$$

$$\Delta \mapsto a_1^6 a_6$$

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Investigate notion of modular forms in a more geometric manner.



$\omega = c^* \Omega^1_{E/S}$   
 line bdl. of inv. 1-forms along the fibers.

then

modular form of wt.  $k$   
 = section of  $\omega^{\otimes k}$  compatible with maps of families of curves.

ex  $\mathbb{C}/\Lambda$   $\omega$  - trivialize using  $dz$ ;  
 section of  $\omega^{\otimes k}$  :  $f(\Lambda) dz^k$ ; compatible with all maps :  $f(t\Lambda) (dtz)^k = f(\Lambda) dz^k$ ,  
 or  $f(t\Lambda) = t^{-k} f(\Lambda)$ . "

← reproduces

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

- gives eigenvectors for scaling in purely algebraic case.