Suppose $R$ = commutative ring

Universal things I can do :

given $x \in R$      square
add $1$
combinations
$\vdots$

Let $V$ = free abelian gp generated by $x$

$\rightsquigarrow$ map $V \to R$ extends to $\underset{\text{(comm. ring)}}{} \operatorname{Sym}(V) \longrightarrow R$

$$y \longmapsto \text{universal expression in } x \qquad (y \in \operatorname{Sym}(V))$$

$X$ = Spectrum

$\operatorname{Sym}^k X = X \wedge \dots \wedge X / \Sigma_k$      $k$- sym power of $X$

this doesn't work as well as it does in algebra , eg.

$\operatorname{Sym}^k(\mathbb{Z}) = \mathbb{Z}$    b/c it's the unit in the tensor structure

but
$\operatorname{Sym}^k(S^0) = \Sigma^\infty (B\Sigma_k)_+$

b/c acts freely ,    acts trivially

( $B\Sigma_k$ = classifying space for the symmetric gp.
$\Sigma^\infty$ : look at all the suspensions
$+$ : add base pt. )

We'll work over the ground ring spectrum K-theory , + p-complete everything.

Call it $K_p$.

Suppose we have $S^0 \xrightarrow{x} R$ comm. $K_p$-algebra , $x \in \pi_0 R$

look at

$$\begin{array}{ccc} \operatorname{Sym}^p(S^0) & \longrightarrow & \operatorname{Sym}^p_{K_p}(R) \\ \downarrow & & \downarrow \\ [\text{illegible}] & & [\text{illegible}] \end{array}$$

so, given a map $S^0 \longrightarrow (K \wedge (B\Sigma_p)_+)^\wedge_p$

find
$$\begin{tikzcd} S^0 \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha(x)"'] & (K \wedge (B\Sigma_p)_+)^\wedge_p \arrow[r] & R \end{tikzcd}$$

Computations:

$$\pi_0 (K \wedge (B\Sigma_p)_+)^\wedge_p \underset{\substack{\text{univ.}\\ \text{coeff.}\\ \text{thm}}}{=} \mathrm{Hom}(\underbrace{K_p^0(B\Sigma_p)}_{\substack{\text{we have to}\\ \text{calculate this}\\ \text{(want f.g.)}}}, \mathbb{Z}_p)$$

<u>thm</u> (Atiyah)

$G$ cpt Lie gp, $K^0(BG) = R[G]^\wedge_I$
representation ring of $G$ completed at augmentation ideal.

$$\Rightarrow K_p^0(B\Sigma_p) = \underset{\substack{\text{trivial}\\ \text{representation}\\ \mathbb{1}}}{\mathbb{Z}_p} \oplus \underset{\substack{\text{permutation}\\ \text{repr. of order } p\\ \text{(permuting set of } p \text{ elts)}\\ \rho}}{\mathbb{Z}_p} .$$

Let $\psi, \vartheta \in \pi_0(K \wedge (B\Sigma_p)_+) = \mathbb{Z}_p \oplus \mathbb{Z}_p$ generated by $\psi, \vartheta$

$\vartheta(\rho) = 1 \qquad \psi(\rho) = 0$

$\vartheta(\mathbb{1}) = 0 \qquad \psi(\mathbb{1}) = 1$

$R$ a $K_p$-algebra commutative

$x \in \pi_0 R \rightsquigarrow \vartheta(x), \psi(x) \in \pi_0 R$
universal construction
(want rel. between these and a poly of deg $p$)

$$\begin{tikzcd} \overbrace{S^0 \wedge \ldots \wedge S^0}^{p} = S^0 = B\Sigma_{1+} \arrow[d] \arrow[rr, "x \wedge \ldots \wedge x"] & & R \wedge \ldots \wedge R \arrow[d] \\ S^0 \wedge \ldots \wedge S^0 / \Sigma_p \quad B\Sigma_{p+} \arrow[r] & & \mathrm{Sym}_K(R) \arrow[d] \\ & & R \end{tikzcd}$$

— corresponds to the inclusion of the units