

Simplicial objects :

Δ = (skeleton) category of finite ordered sets

Objects : $[n] = \{0 < 1 < \dots < n\}$, $n \geq 0$.

$\Delta \rightarrow \text{Spaces}$

$[n] \mapsto \Delta^n = \text{standard } n\text{-simplex}$

$\downarrow \Theta \quad \downarrow \Theta_n = \text{affine linear}$

$[m] \mapsto \Delta^m$

A simplicial object in cat. \mathcal{C} is a functor

$$\Delta^{op} \rightarrow \mathcal{C}$$

ex X space, $[n] \mapsto \text{Map}(\Delta^n, X) = \text{Sim}_n(X)$ is a simplicial set.

Cosimplicial obj. are functors

$$\Delta \rightarrow \mathcal{C}$$

ex $[n] \mapsto \Delta^n$ cosimpl. space Δ^{\sim} .

If $P.$ is a simpl. A_0 -ring spectrum then $A_0(P., F)$ is a cosimplicial space.

Def If X' is a cosimpl. space, then

$Tot(X') =$ Space of maps Δ^i to X'

$$\subset \coprod_{n \geq 0} \text{Map}(\Delta^n, X^n)$$

An elem. of $Tot(X')$ consists of a pt. $x_0 \in X^0$, a path in X^1 from $d^0 x_0$ to $d^1 x_0$, ---

Cannot necessarily form

$$X' \rightsquigarrow \pi_k X' \text{ cosimpl. ab. grp}$$

— base-point problem.

Suppose we start with $x_0 \in X^0$ and a path in X^1 from $d^0 x_0$ to $d^1 x_0$. Get a map of cosimpl. spaces

$$sk_1 \Delta^1 \longrightarrow X'$$

Since $sk_1 \Delta^1$ is connected, we get

$\pi_k(X^n, x)$. Assuming $\pi_1(X^n, x)$ acts trivially
on $\pi_k(X^n, x)$, $k \geq 0$, we can form

$\pi_k(X^n, x)$ — a cosimpl. ab. grp

Discuss algebra from last time in more detail.

$A \xrightarrow{\alpha} B$ map of comm. rings.

$P \rightarrow B$ simpl. resolution with each P_n free associative A -algebra.

M B -module.

Then get a cosimpl. A -module

$$\text{Der}_A(P, M)$$

with assoc. cohomology groups the derived functors of $\text{Der}_A(-, M)$.

Derivations: let A be a comm. ring and B a (not necessarily comm.) A -alg., i.e. a ring homomorphism

$$A \xrightarrow{\alpha} B$$

s.t. $\alpha(A)$ is contained in the center of B . Let M be a B -bimodule s.t. for all $a \in A$, $am = ma$.

An A -linear derivation of B into M is an A -linear map

$$B \xrightarrow{D} M$$

s.t. $D(b_1 b_2) = D(b_1) b_2 + b_1 D(b_2)$. Then B is a universal A -linear derivation of B into a B -bimodule:

$$0 \rightarrow I \rightarrow B \otimes_A B \xrightarrow{\mu} B \rightarrow 0$$

$B \rightarrow I$ a derivation

$$b \mapsto b \otimes 1 - 1 \otimes b$$

Universal:

$$\text{Der}_A(B, M) \cong \text{Hom}_{B-B}(I, M)$$

Given $P \rightarrow B$ a simpl. res., then

$$I_1 = \ker(P \otimes_A P \xrightarrow{\mu} P)$$

is a P -bimodule, and as cosimpl. A -modules

$$\text{Der}_A(P, M) \cong \text{Hom}_{P-P}(I_1, M)$$

Since M is a P -bimodule via the augmentation $P \rightarrow B$, we have

$$\begin{aligned} \text{Hom}_{P-P} (I, M) \\ \approx \text{Hom}_{B-B} \left(\underbrace{B \otimes_{P} I \otimes_{P} B}_{D_{B/A}}, M \right) \end{aligned}$$

where

$$D_{B/A} := B \otimes_{P} I \otimes_{P} B$$

is Quillen's assoc. alg. homology object of B/A ;

$$B^e := B \otimes_A B^{op} \text{ enveloping algebra}$$

$$B\text{-bimodule} = \text{left } B^e\text{-module.}$$

Suppose B is commutative and bncmb.

Then further

$$\text{Hom}_P (I, M) \approx \text{Hom}_{B^e} (D_{B/A}, M)$$

$$\approx \text{Hom}_B \left(\underbrace{B \otimes_{B^e} D_{B/A}}_{\wedge_{B/A}}, M \right)$$

Commutative algebra analog:

$A \rightarrow B$, M B -module

$$\text{Der}_A(B, M) := \left\{ \begin{array}{l} B \xrightarrow{D} M \\ D(k_1 b_2) = D(k_1) b_2 + k_1 D(b_2) \\ D(a) = 0 \end{array} \right\}$$

Universal derivation

$$B \xrightarrow{D} B \otimes_{B^c} I = I/I^2 =: \Omega_{B/A}$$

Start with B/A , pick simple res.
 $Q. \rightarrow B$ by free comm. A -algebras.
 Obtain as before the André-Quillen
 homology object

$$L_{B/A} = \Omega_{Q./A} \otimes_{Q.} B$$

s.t. the derived functors of the
 derivations for comm. algebras B

$$\text{Der}_A^s(B, M) = H^s(\text{Hom}_B(L_{B/A}, M))$$

Thm Suppose B is flat over A . Then
 if $L_{B/A}$ is acyclic, $\Lambda_{B/A}$ is acyclic.

Before we give the proof ---

ex Suppose $A = \mathbb{F}_p$, $\varphi: B \xrightarrow{\sim} B$, $\varphi(x) = x^p$

Then $L_{B/A}$ is acyclic, so by the thm.

$\wedge_{B/A}$ is acyclic, so derived functors of assoc. alg. derivations vanish:

$$\text{Der}_A^s(B, M) = 0, \quad s \geq 0.$$

To see that $L_{B/A}$ is acyclic:

$$\begin{array}{ccc} Q_n & \xrightarrow{d_n} & Q_{n-1} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varphi} & B \end{array} \quad Q_n = A[x_i] \xrightarrow{\varphi} A[x_i^p]$$

Since $d(x_i^p) = p x_i^{p-1} dx_i = 0$, the map on $L_{B/A}$ induced by φ is zero. But it is also chain-homotopic to an iso.

Therefore, $L_{B/A}$ is acyclic. //

Prop (Quillen) If B is flat over A ,

$$H_k(\wedge_{B/A}) = \text{Tor}_{k+1}^{B^c}(B, B).$$

proof Since $P \rightarrow B$ is a res. of B

by proj. A -modules, and since B/A is flat, $P \otimes_A P \rightarrow B^e$ is again a res. by proj. P -modules.

$$\begin{array}{ccccc}
 0 \rightarrow I & \rightarrow & P \otimes_A P & \xrightarrow{\mu} & P \rightarrow 0 \\
 & & \downarrow \sim & & \downarrow \sim \\
 0 \rightarrow I^B & \rightarrow & B \otimes_A B & \xrightarrow{\mu} & B \rightarrow 0
 \end{array}$$

so $I \rightarrow I^B$ is a res. by proj. P^e -mod. So get res. by proj. B^e -mod

$$B^e \otimes_{P^e} I \xrightarrow{\sim} I_B$$

Now

$$\wedge_{B/A} := B \otimes_{B^e} (B^e \otimes_{P^e} I)$$

so

$$H_i(\wedge_{B/A}) = \text{Tor}_{B^e}^i(I_B, B)$$

The prop. follows from the long-exact seq. assoc. w. the s.e.s. of B^e -mod.

$$0 \rightarrow I_B \rightarrow B^e \xrightarrow{\mu} B \rightarrow 0$$

"