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(1)
Hopkins Miller Seminar

Algebraic Stacks and why Homotopy theorists are interested in them

$$\{\text{Schemes over } \mathbb{C}\} \xrightarrow[\text{embedding}]{\text{Yoneda}} \left\{\begin{matrix}\text{contravar. functors} \\ \text{Schemes} \to \text{Sets}\end{matrix}\right\}$$

just for our comfort

$$X \longmapsto \mathrm{Hom}(-, X)$$

Lemma (Yoneda): The Yoneda embedding is fully faithful.

Observation (Grothendieck): This is usefull.

Example: Functor of smooth cubic plane curves

$$F: S \mapsto \left\{ \text{closed subschemes } Y \text{ of } S \times \mathbb{P}^2 \;\middle|\; \begin{matrix} Y \text{ is flat over } S \text{ \& each fibre is a smooth} \\ \text{cubic in } \mathbb{P}^2 \end{matrix} \right\}$$

"family of cubic curves parametrized by S"

functor by pullback.

This functor F is representable.

$\mathbb{C}^{10}$ ← homogeneous cubics in 3 variables

$\cup$

$\mathbb{C}^{10} \setminus \{0\} \longrightarrow \mathbb{P}^9$ represents all cubic curves on the plane

But not all of these are smooth

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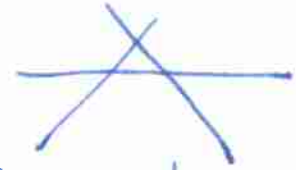
e.g. xyz  zero set — pts are singular

$U \subset \mathbb{P}^9$ subset corresponding to smooth cubics

(complement of hyperplane $\Rightarrow$ Zariski open)

U repr. F,

$F(S) = \mathrm{Hom}(S, U)$

in particular there is a universal cubic curve

$Y \subseteq U \times \mathbb{P}^2$.

[elliptic curve usually means genus one curve with basepoint, but we are going to be sloppy & ignore bpt & say ell. curve = gen. one curve]

[smooth cub. curves in plane are ell. curves]

Let's look at another moduli problem:

$$F(S) = \left\{ \begin{array}{c} E \\ \downarrow \pi \\ S \end{array} \;\middle|\; \begin{array}{l} \pi \text{ is proper, flat,} \\ \text{geom. fibres are connected} \\ \text{smooth genus 1 curves} \end{array} \right\} \Big/ \begin{array}{l} \text{isoms} \\ \text{over } S \end{array}$$