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# On the construction of the $EO_n$ 's

André

①

$$\left( \begin{array}{l} \text{reduce} \\ \text{mod} \\ M \end{array} \right) \cdot \pi_0 : A_{\infty}^{LT} \longrightarrow \mathcal{Fg}_{\text{sep. closed}}^{\text{or}}$$

Seminar  
on the  
Hopkins  
Miller thm

is an equivalence of topological categories, i.e.

- essentially surjective
  - fully faithful in topological sense :
- } have shown this in Vigneri's talk

$$A_{\infty}^{LT}(E, F) \longrightarrow \mathcal{Fg}((\mathbb{R}_2, \Gamma_2), (\mathbb{R}_1, \Gamma_1))$$

is a homotopy equivalence.

↳ discrete topology

$$E_2 \in A_{\infty}^{LT} \longmapsto (F_4, \Gamma_2)$$

$$G = \tau^* \times \mathbb{Z}/12 \quad \text{⌚}$$

↑ binary tetrahedral group

tetrahedral group (order 12)  $\subset SO(3)$

look at preimage in univ. cover of  $SO(3)$

i.e. in  $SU(2)$ ,

this is a central extension (universal one)  
there is

an action of  $\mathbb{Z}/2$  on the tetrahedral group

$$1 \rightarrow A \rightarrow \overline{G} \rightarrow G \rightarrow 1$$

corresponding to the outer autom we get by exchanging vertices & faces.  $\rightsquigarrow \times$

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Also write  $G$  for the category  $\mathcal{G}_G$ .

We have  $G \rightarrow \text{Fig}^{\text{op}}$

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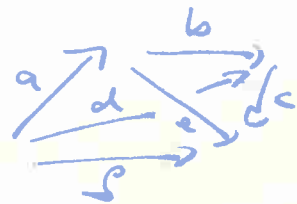
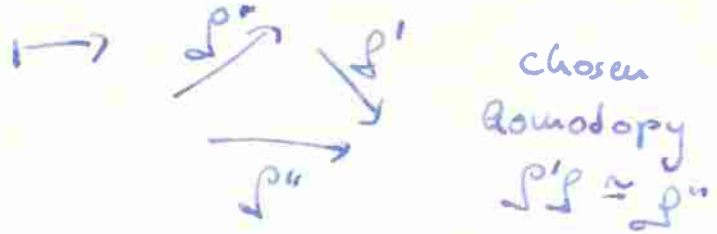
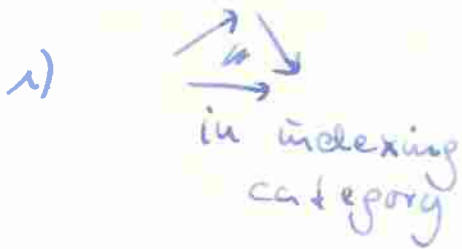
question: can we lift this?



Once we lift it (for a good model of  $E_2$  that has strict action of  $G$ ).

We can form  $EO_2 = E_2^{hG} = F(EG_+, E_2)^G = \frac{\text{Equiv } E_2}{G}$

① Relax notion of functor

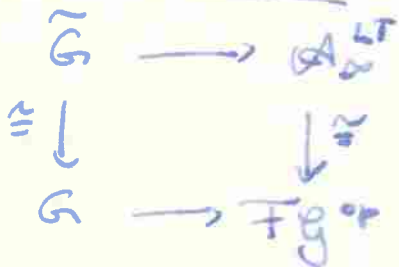


$$\begin{matrix} ea & \cong & f \\ 21 & \text{diagonal} & k \\ cba & \cong & ccb \end{matrix}$$

3) ... cube ...

with allow us to def some notion of  $E^{hG}$  extra piece

② second idea:



$\tilde{G} = \left\{ \begin{array}{l} 1 \text{ object} \\ \text{morphisms} = A_{\infty}^{LT}(E_2, E_2) \\ \text{that map to } G \\ \text{i.e. has 24 connected components} \dots \end{array} \right\}$

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(3)

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$$EO_2 = \underset{\hat{G}}{\text{colim}} E_2$$

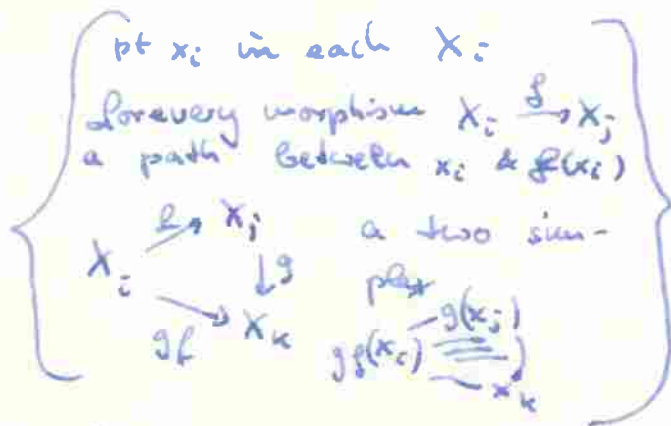
(3) of course the best thing would be to  
really have an action on the nose  
this is rather difficult but convenient

For later.

Recall:  $\text{colim}$  for spaces

$J \rightarrow \text{Spaces}$   
↑ indexing category

want  
a point in  $\text{colim } X_i$  is



Now reformulate this for  $X_i = \text{spectra}$  &  
 $\mathcal{C} \supset (\hat{=} \hat{G})$  a topological category:

$$Y^0 = \prod_{i \in J} X_i$$

$$Y^1 = \prod_{i, j \in J} X_i \times \text{Hom}(X_i, X_j) \times J(i, j)$$

$$Y^{u-1} = \prod_{i_0, \dots, i_u} X_{i_u} \times J(i_0, i_1) \times J(i_1, i_2) \times \dots \times J(i_{u-1}, i_u)$$

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④

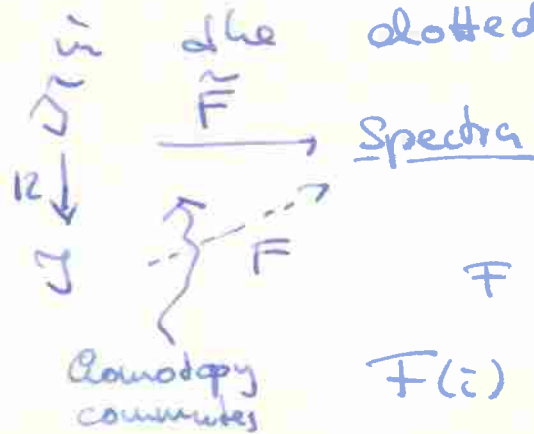
Seminar on  
the Hopkins  
Miller class

cosimplicial spectrum,

$\text{Tot}(-)$  gives back  $\Delta$ 's  
with all the nice compatibility  
conditions

$$\text{Tot}(\mathcal{F}) =: \text{colim}_{\leftarrow J} X_i$$

At this point it becomes easy to  
fill in the dotted arrow above



$\mathcal{F}$  is defined by

$$\mathcal{F}(i) = \text{colim}_{\leftarrow \substack{(i \rightarrow j) \text{ all} \\ \text{arrows} \\ \in (i \downarrow J)}} \tilde{\mathcal{F}}(j)$$

$$\tilde{\mathcal{F}}(i)$$

$i = \text{initial object}$

Where are we and why?

We are ultimately interested in  $\pi_* \mathbb{S}^0$ , or say  $\pi_* \mathbb{S}^0(p)$ . But the  $p$ -local stable homotopy category is already much more complicated than the rational one (latter  $\cong$  graded  $\mathbb{Q}$  vs'  $\rightarrow$  Mike Hill)

There is a theorem, the thick subcategory theorem that is making the sense in which this category is more complicated precise: it tells us that for each  $p$  there is a tower of non-trivial Bousfield localizations of  $\mathcal{S}_{(p)}^{fin}$  (finite  $p$ -local spectra). Think of Bousfield localization at a homology theory  $E_n(-)$  as

$$" \mathcal{S}_{(p)}^{fin} / E\text{-acyclic spectra (i.e. } x \text{ s.t. } E_n(x) = 0) "$$

also denoted  $\langle E \rangle$ .

Fact:  $L_E = L_F \iff \langle E \rangle = \langle F \rangle$ ,  $\langle E \rangle \leq \langle F \rangle \implies L_E L_F = L_F$

Thick subcategory theorem:  $\mathcal{S}_{(p)}^{fin}$  has a filtration  $L_E \rightarrow L_F$  (Hopkins, Smith)

$$(*) \mathcal{S}_{(p)}^{fin} = \mathcal{E}^0 \supseteq \mathcal{E}^1 \supseteq \mathcal{E}^2 \dots \supseteq \mathcal{E}^n \supseteq \dots$$

$\uparrow$   
=  $K(n-1)$ -acyclics

s.t. for any spectrum  $E$ ,  $\langle E \rangle = \mathcal{E}^n$  for some  $n$ .

Note:  $K(n) = n^{\text{th}}$  Morava  $K$ -theory, constructed by Jardine & Sullivan, using cobordism of  $m$ -fs w/ singularities, coefficients are  $\mathbb{F}_p[u_n^{\pm 1}]$  ( $\rightsquigarrow$  Kjeilek's first talk).  $|u_n| = 2p^n - 2$  carry Honda spl.

- The inclusions are a non-trivial statement, only true for finite spectra "universal"
  - On finite spectra  $\langle K(n) \rangle = \langle E(n) \rangle$ ,  $E(n) = \text{Johnson}$  "high n spl"  
Wilson spectra def. by LEFT &
- $$E(n)_+ = \mathbb{Z}_{(p)}[u_1, \dots, u_n^{\pm 1}]$$

(\*) This is called "the chromatic filtration of the stable homotopy category".

- For big enough  $p$ , there are alg. models for  $\mathcal{S}_{(p)}^{fin}$ .
- $X \in \mathcal{E}^n - \mathcal{E}^{n+1}$  is called fixed n - 1 - "of type  $n$ ".

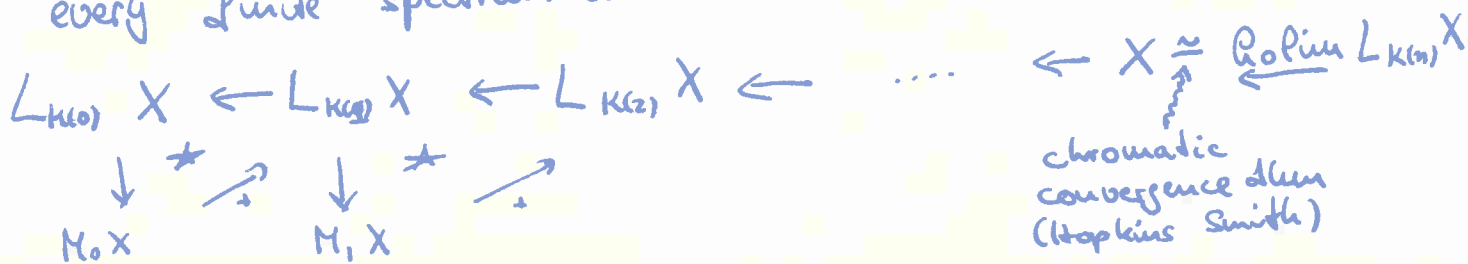
Since  $\pi_* \mathbb{S}_{(p)}^0$  is too difficult for us to understand, we can try our luck in the localized categories and look at  $\pi_*(L_E X)$ . ( $L_E$  <sup>(induced by the)</sup> ~~is~~ <sup>(L-functorial)</sup> fibration replacement in new model str (BL keeps w.fib, extends w.eqs. & therefore has fewer ~~maps~~) or  $\mathcal{S} \xrightarrow[\mathcal{R}]{\mathcal{L}} \mathcal{S}_E$  has right adjoint,  $L_E := R \circ \mathcal{L}$ .) (careful: it is not clear whether  $\mathcal{S}_{(p)}^{\text{fin}} \subset \mathcal{S}_{(p)}^{\text{fin}}$  is full  $\leadsto$  telescope conj. (?) )

The thick subcategory theorem tells us that  $L_{K(0)}$  should be the easiest to understand (indeed  $K(0) = H(-; \mathbb{Q})$ ), then  $L_{K(2)}$  and so on. ( $K(n) = H(-, \mathbb{Z}/p^n)$ )

~~How far are people today~~

In other words: (thick subc. thm)

every finite spectrum admits a tower



apply  $\pi_*$  to get an exact couple, the corresponding spectral sequence is called the geometric chromatic spectral sequence, it converges to the (geometric) chromatic filtration of the stable homotopy groups of  $X$

$$\dots F^{n-1} \subseteq F^n = \text{ker}(\pi_* X \rightarrow \pi_* L_{K(n)} X) \in \pi_* X$$

How far are we today?

$\pi_* L_{K(0)} \mathbb{S}^0$  ✓ well understood

$\pi_* L_{K(2)} \mathbb{S}^0$ : ~~this~~ is ~~computed~~: It is the image of  $J$ .

Therefore there is a geometric definition of these elements.

Recall from Mike Hopkins' table that it is computed at

odd primes using some sort of spectral sequence that had to do with  $K_p^\wedge$ , and that at  $p=2$ ,  $KO_2^\wedge$  was a better starting point than  $K_2^\wedge$  (compare also Adams original paper on the image of  $\eta$  IV).

As of today, people are still trying to get a good understanding of the second chromatic localization

$L_{K(2)} \mathcal{S}^0$ . For this purpose,

$$E_2 := E_{F_4, \Gamma = \text{Honda}} \text{ sgl}$$

will play the rôle of  $K_p^\wedge$ -theory, and  $EO_2$  will play the rôle of  $KO_2^\wedge$  (at primes 2 and 3 ~~and~~).

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More precisely: To compute  $L_{K(n)} \mathcal{S}^0$ ,

one uses the ~~Ass~~  $K(n)$ -local  $E_n$ -Adams-

spectral sequence (discussed in the Appendix of Dev. & Hopkins's paper on City fixed points spectra for closed subgroups of the Morava stabilizer group),

~~Setup is like traditional~~ this is the  $E_n$ -ASS in the  $K(n)$ -local category, convergence follows from the fact that any  $K(n)$ -local spectrum  $X$  is  $E_n$ -nilpotent

def:  $F_*$ -local  $E$ -nilp spectra smallest class  $\mathcal{C}$  of  $F_*$ -local spectra s.t.

- i)  $L_F E \in \mathcal{C}$
- ii)  $L_F(N \wedge X) \in \mathcal{C}$  whenever  $N \in \mathcal{C}$
- iii)  $\mathcal{C}$  is closed under retracts & cofibres

$$E_n = E_{\mathbb{F}_p^n}, \text{ Honda's gl}$$

Don't confuse with  $E(n)$ !

Devine and Hopkins show that this spectral sequence is actually a homotopy fixed point spectral sequence (in a continuous sense) for  $\mathbb{Z}$

$$L_{K(n)} \mathbb{S}^0 \simeq E_n^{hG_n} \text{ full Morava stabilizer group} \\ \times \text{action of Galois group.}$$

Its  $E_2$ -term is the same as that of the chromatic spectral sequence

htpy fixed pts s.s.  $H_c^*(G_n, E_{n*}) \Rightarrow \Pi_* L_{K(n)} \mathbb{S}^0$

$K(n)$ -local  $E_n$  ASS  $\text{Ext}_{E_n * E_n}^{||R} (E_{n*}, E_{n*} \mathbb{S}^0) \xrightarrow{\text{same s.s.}} \Pi_* L_{K(n)} \mathbb{S}^0$

$\underbrace{\hspace{10em}}_{\cong E_{n*} \mathbb{S}^0}$

THIS MIGHT BE NOT TRUE, BUT THE TELESCOPE CONJECTURE?

and I believe that that is not by coincidence, Ravenel, red book p. 80 (top) makes a sort of mysterious remark that sounds like their E.s.s. could indeed be the same as the ANSS for  $L_{K(n)} \mathbb{S}^0$ . Then we would have a picture of s.s. looking like this

$$H_c^*(G_n, E_{n*}) \Rightarrow \Pi_* L_{K(n)} \mathbb{S}^0 \\ \Downarrow \\ \text{Ext}_{\mathbb{B}P_* \mathbb{B}P}^{(\mathbb{B}P_*, \mathbb{B}P_*)} \Rightarrow \Pi_* \mathbb{S}^0$$

However, historically, the algebraic side was there first.

and possibly the ~~at~~ left vertical spectral sequence is the same as the chromatic spectral sequence (?).