520 Homework 1

Note Title 9/1/2004

PROBLEM 1: Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. Then $\{(f, \mathbb{R})\}$ is an atlas on $\mathbb{R}$ with one chart. Assume $g : \mathbb{R} \to \mathbb{R}$ is likewise a homeomorphism.

A. Show that the atlas $\{(f, \mathbb{R})\}$ is equivalent to $\{(g, \mathbb{R})\}$ if and only if

$$f \circ g^{-1} \quad \text{and} \quad g \circ f^{-1} \quad \text{are smooth}$$

B. Conclude that there are infinitely many inequivalent smooth structures on $\mathbb{R}$.

C. Show that the manifolds $\{(f, \mathbb{R})\}$ and $\{(g, \mathbb{R})\}$ are diffeomorphic.

It is a deep theorem that all smooth structures on $\mathbb{R}^n$ are diffeomorphic for fixed $n \neq 4$.

PROBLEM 2: Show that the cross product of two manifolds is a manifold.

PROBLEM 3: Show that $X \times Y$ is diffeomorphic to $Y \times X$ for all manifolds X, Y.

PROBLEM 4: Show that matrix multiplication defines a smooth map

$$\mu : GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$$

DEF: Any group $\mathcal{G}$ which is a manifold and for which the group operation is smooth is called a LIE GROUP

PROBLEM 5: Show that, for fixed $A \in GL_n(\mathbb{R})$, the map

$$\mu_A : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$$
$$B \mapsto AB$$

is a diffeomorphism.

REMARK: The same proof yields a better result: for any Lie group $\mathcal{G}$ and $g_0 \in \mathcal{G}$ the map $g \mapsto g_0 \cdot g$ is a diffeomorphism.