Lecture 23: Volume in dimension 2,3,4,5,6,...

Today what we will do is flex our muscles a bit and show how to compute volumes of objects in arbitrary dimensions. "Volume" in dimension one clearly means "length"; in dimension two it means "area"; in dimension three it means, well, volume. What about dimensions four and higher? Does it really matter? [Here comes an aside...] Sure! Space-time is four-dimensional and notions of volume are not without relevance. More recent theories of matter involve working in spaces of dimensions 8 or higher. On a more practical level, any mechanical system without friction [a good model for, say, the solar system or for a low-viscosity fluid] evolves in an abstract *phase space* (a space that, loosely speaking, keeps track of positions and velocities). In Hamiltonian dynamics, one of the principal theorems one learns is the Liouville theorem, which states that the dynamics on the phase space conserves volumes. Though that may have little meaning to you, the point is that volumes in abstract n-dimensional spaces are really quite important.

So let's think. Hmmmmm. Hummmm. Grummm. Aha! You can compute the volume of a cube in dimension n quite easily, right? Let's look at a cube of side-length r. Then the n-dimensional volume of this "hypercube" is r^n — the obvious generalization of "length times width times height". Note that 1-d volume is length and 2-d volume is area.

Now what about a round ball of radius r? Here, things are not so easy. The 1-d ball of radius r is a line segment of length $2r$. The 2-d ball of radius r is a disc of area πr^2. The 3-d ball of radius r has volume $\frac{4}{3}\pi r^3$. Interesting. There is still an r^n term, but now there is a coefficient in front of it. How can we compute this? Will it always be of the form "some fraction times π"? Could it ever have square roots in it? Or something weirder?

Denote by β_n the volume of the n-dimensional ball of radius 1. It is a [pretty obvious] truth that the volume of the n-dimensional ball of radius r is $\beta_n r^n$. Thus we have $\beta_1 = 2$, $\beta_2 = \pi$, and $\beta_3 = \frac{4}{3}\pi$.

To compute β_n, we will use what is called the "disk method" — slice the n-ball by parallel planes of dimension $n-1$. For a 2-d ball, these slices are line segments (i.e., 1-d balls); for a 3-d ball, these slices are 2-d balls; likewise, for a 4-d ball, the slices are 3-d balls.

But note what happens to the radii of the slices: they start at zero, increase until they hit radius 1, then decrease back to zero. Indeed, if we denote the n^{th} coordinate axis by x_n, then the slice on the n-ball along the plane $x_n = t$ is an $n-1$ ball of radius $\sqrt{1-t^2}$ [you should check that this works in lower dimensions!].

One integrates with respect to t (the slicing direction) and sums up the volumes of the slices. In our case this becomes

$$\beta_n = \int_{-1}^{1} [\text{volume of } n-1 \text{ ball of radius } t]dt$$

But the volume of the $n-1$ ball of radius r is $\beta_{n-1}r^{n-1}$. Thus we obtain the inductive formula:

$$\beta_n = \int_{-1}^{1} (\sqrt{1-t^2})^{n-1}\beta_{n-1}dt = \beta_{n-1}\int_{-1}^{1}(\sqrt{1-t^2})^{n-1}dt$$

where we can pull out the β_{n-1} term since it is a constant (doesn't depend on t). This is an "inductive" formula — it does not give us a numerical value for β_n, but if we know what β_{n-1} is, we can figure it out. Assuming we can do that tricky little integral there...

As for that integral, one can use a trig substitution $t = \sin\theta$ to get that

$$\int_{-1}^{1}(\sqrt{1-t^2})^{n-1}dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n\theta\, d\theta$$

Doing integration by parts twice, one gets the formula from the integral tables:

$$\int \cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta\sin\theta}{n} + \frac{n-1}{n}\int\cos^{n-2}\theta\, d\theta$$

Let us denote by c_n the numerical value

$$c_n := \int_{-1}^{1}(\sqrt{1-t^2})^{n-1}dt$$

so that $\beta_n = c_n\beta_{n-1}$. Plugging in our limits of integration to the above formula yields

$$c_n = \left.\frac{\cos^{n-1}\theta\sin\theta}{n}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{n-1}{n}c_{n-2} = \frac{n-1}{n}c_{n-2}$$

since the former terms are zero for all $n \geq 2$. So we are left with the two inductive formulæ

$$\beta_n = c_n\beta_{n-1} \quad ; \quad c_n = \frac{n-1}{n}c_{n-2}$$

What are the "seeds" of this induction? We know that $\beta_0 = 2$ and $\beta_1 = \pi$. We can also compute the first few terms of c_n explicitly:

$$c_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^0\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

$$c_1 = \int_{-1}^{1}(\sqrt{1-t^2})^{1-1}dt = \int_{-1}^{1}dt = 1+1 = 2$$

(I've chosen the form of c_n carefully to make the integrals easy!)

So, now we can start cranking out a table:

$$\begin{aligned} c_2 &= \frac{2-1}{2}c_0 = \frac{\pi}{2} \\ c_3 &= \frac{3-1}{3}c_1 = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} c_4 &= \frac{4-1}{4} c_2 = \frac{3\pi}{8} \\ c_5 &= \frac{5-1}{5} c_3 = \frac{16}{15} \\ c_6 &= \frac{6-1}{6} c_4 = \frac{15\pi}{48} \end{aligned}$$

The volumes of the unit n-balls are thus

$$\begin{aligned} \beta_2 &= c_2\beta_1 = \pi \\ \beta_3 &= c_3\beta_2 = \frac{4\pi}{3} \\ \beta_4 &= c_4\beta_3 = \frac{\pi^2}{2} \\ \beta_5 &= c_5\beta_4 = \frac{8\pi^2}{15} \\ \beta_6 &= c_6\beta_5 = \frac{\pi^3}{6} \end{aligned}$$

There are some very nice patterns here!

This is meant to show you how calculus can compute things for you that you cannot see, feel, taste, or touch: the mind triumphant over matter. It also shows a good example of an inductive solution to a nontrivial problem.

Now, let's have some fun. Suppose I tell you that with a little bit more work, you could prove that

$$\beta_{2n} = \frac{\pi^n}{n!} \text{ and } \beta_{2n+1} = \frac{\pi^n n! 2^{2n+1}}{(2n+1)!}$$

That looks messy, but is really very natural — there must be an even/odd dichotomy as seen experimentally above.

Here are a few questions to think about tonight, after dinner, while sipping your cup of green tea.

1. Compare the volume of the unit n-ball to the volume of the unit n-cube (always 1). It is sometimes bigger and sometimes smaller....
2. What is the limit of the volume of the unit n-ball as n goes to infinity? The limit does exist, and it is zero! What does this mean? The unit cube stays the same volume, but the unit ball "looks" thinner and thinner in higher dimensions. This really is telling you something about what very high dimensional spaces look like — the "corners" of the cube take up more space than the "ball" in the middle. Consider this well: it is beautiful!
3. What would happen if you look at the sequence β_n and graph them? You would get a unique maximum. Where? It is in dimension five, where $\beta_5 \approx 5.263789009....$ Why is this? I do not know the answer to this question...