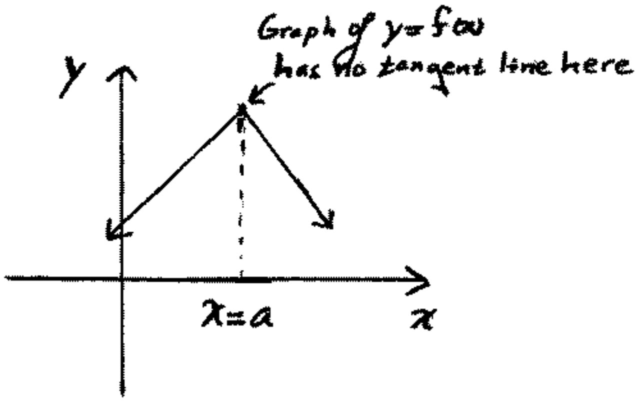
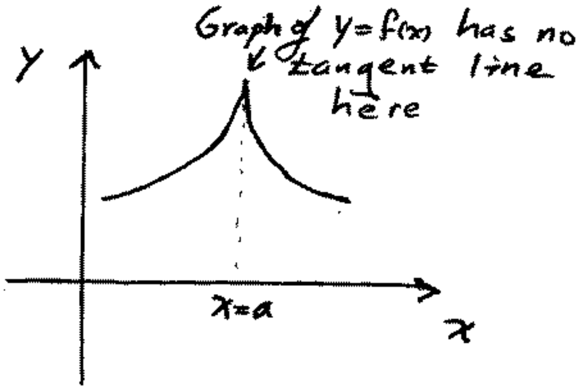
Section 1.5: Differentiability and Continuity

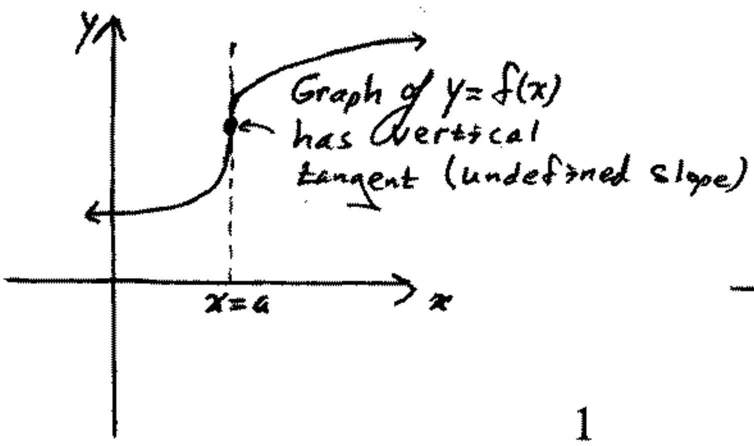
Goal: To connect the ideas of limits, continuity, and differentiability. To see what the graphs of differentiable functions look like.

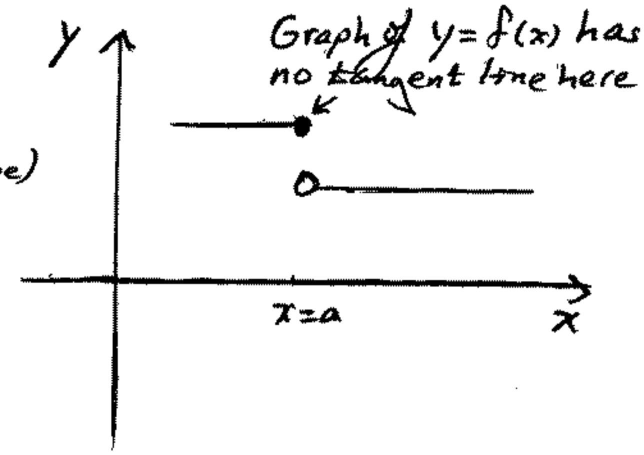
Definition: A function $f(x)$ is *differentiable* at $x = a$ if $f'(a)$ exists (that is, if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists).

Look at the following examples, which show how a function can fail to be differentiable at a point $x = a$.



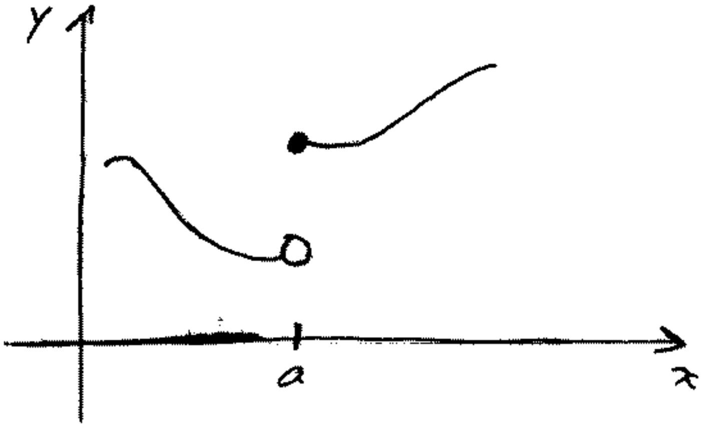






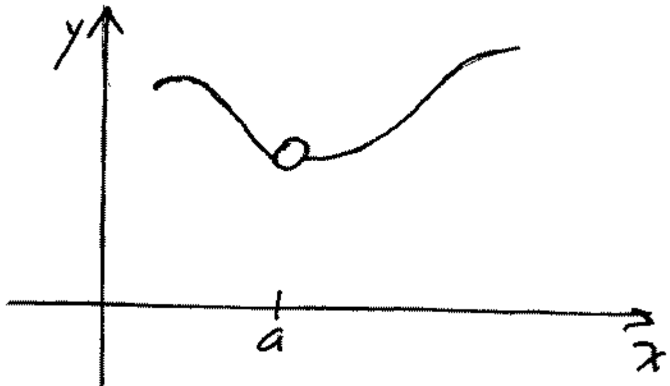
<u>Definition</u>: A function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Intuitively, a function is continuous at $x = a$ if the graph of $f(x)$ has no breaks, jumps, or holes as it passes through the point $(a, f(a))$. Look at the following examples of functions which are not continuous at $x = a$.



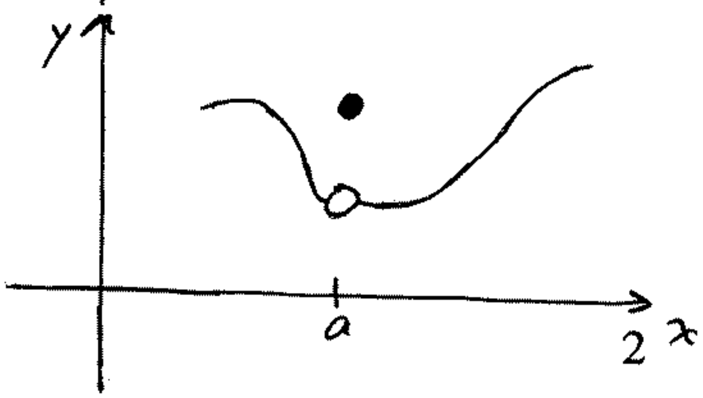
$y = f(x)$ has a jump at $x = a$.

$\lim_{x \to a} f(x)$ does not exist.



$y = f(x)$ has a hole at $x = a$.

$\lim_{x \to a} f(x)$ exists but $f(a)$ is not defined.



$y = f(x)$ has a hole at $x = a$.

$\lim_{x \to a} f(x)$ exists and $f(a)$ exists, but $\lim_{x \to a} f(x) \neq f(a)$.

Connection between differentiability and continuity

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Example: (#16, page 108)

$$f(x) = \begin{cases} x^3 & \text{for } 0 \le x < 1 \\ x & \text{for } 1 \le x \le 2 \end{cases}$$

Determine whether $f(x)$ is continuous and/or differentiable at $x = 1$.

Question 1: Is $f(x)$ continuous at $x = 1$? By the definition of continuity we need to check whether $\lim_{x \to 1} f(x) = f(1)$.

Notice that $f(x)$ is defined differently to the left and to the right of $x = 1$, so we should check the limit from each direction separately:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = 1^3 = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1+} x = 1$$

Hence, $\lim_{x \to 1} f(x) = 1$.

Since $f(1) = 1$, $\lim_{x \to 1} f(x) = 1 = f(1)$.

Therefore $f(x)$ is continuous at $x = 1$.

Question 2: Is $f(x)$ differentiable at $x=1$?

Since $f(x)$ is defined differently to the left and to the right of $x=1$, check the derivative to each side of $x=1$.

To the left of $x=1$:

$$\frac{d}{dx}x^3 = 3x^2 \text{ and } \lim_{x\to 1^-} 3x^2 = 3$$

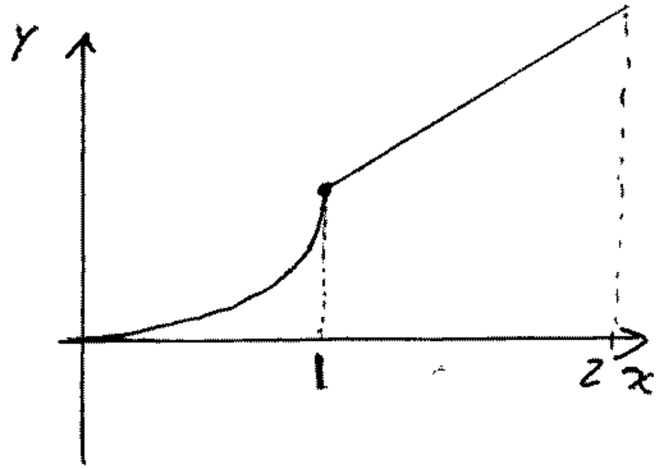
To the right of $x=1$:

$$\frac{d}{dx}x = 1 \text{ and } \lim_{x\to 1^+} 1 = 1$$

So, $f(x)$ has slope 3 as x approaches 1 from the left and a slope of 1 as x approaches 1 from the right. $3 \neq 1$ so that $f(x)$ is *not* differentiable at $x=1$.

Pictorially, we have:

$$y = f(x) = \begin{cases} x^3 & 0 \le x < 1 \\ x & 1 \le x \le 2 \end{cases}$$

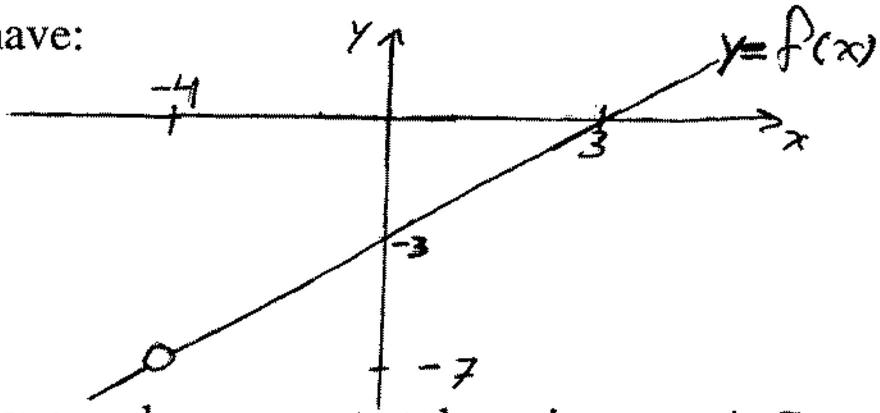


We see that $f(x)$ is continuous but not differentiable at $x=1$.

Example: (#22, page 109)

$$f(x) = \frac{x^2 + x - 12}{x + 4} \quad \text{for } x \neq -4$$

Pictorially, we have:



$f(x)$ is defined everywhere except at the point $x = -4$. Can we fill in the hole in the graph to make the function continuous at $x = -4$?

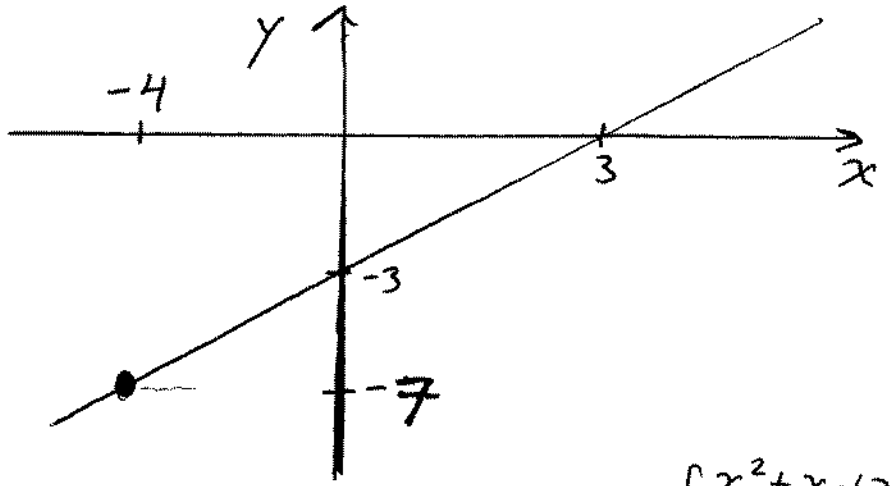
$$\lim_{x \to -4} f(x) = \lim_{x \to -4} \frac{x^2 + x - 12}{x + 4}$$

$$= \lim_{x \to -4} \frac{(x+4)(x-3)}{(x+4)}$$

$$= \lim_{x \to -4} (x - 3)$$

$$= -4 - 3 = -7.$$

So the hole is at the point (–4,–7). If we define $f(-4) = -7$, then we fill in the hole, and $f(x)$ is continuous at $x = -4$.



$$f(x) = \begin{cases} \dfrac{x^2+x-12}{x+4} = x-3 & \text{for } x \neq -4 \\ -7 & \text{for } x = -4 \end{cases}$$

Now $\lim_{x \to -4} f(x) = -7 = f(-4)$.