Section 1.8: The Derivative as a Rate of Change

Goal: Understanding the derivative as a rate of change and applying this to real world problems.

The distance between Champaign and Chicago is about 125 miles. Suppose I drive from Champaign to Chicago in about two and a half hours. How fast did I drive?

Answer: $\frac{125}{2.5} = 54$ So 54 mph.

Question: Did I speed?
Question: What did we just compute? (*Average* speed)
Question: How do we find *exact* speed at a given instant?

Answer: The derivative is a math speedometer.
Specifically,

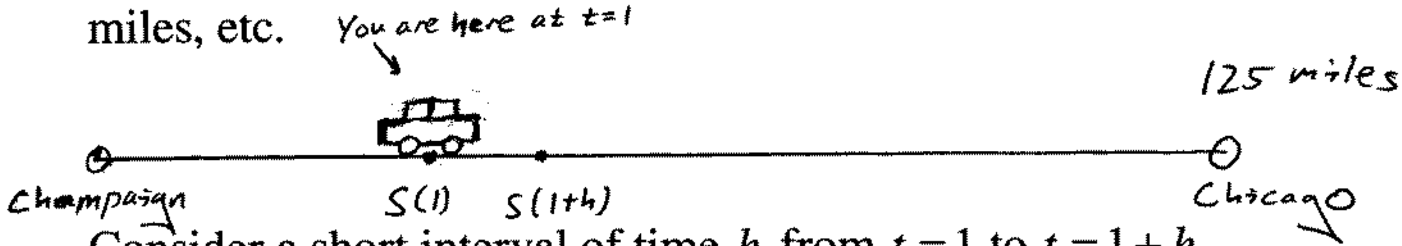
If $s(t) =$ distance with respect to time,
then $s'(t) =$ velocity with respect to time,
and $s''(t) =$ acceleration with respect to time.

Buzz Words:

1) Average rate = $\frac{f(x+h) - f(x)}{h}$ for $h \neq 0$.
2) Velocity, rate, marginal, instantaneous rate all = f'
3) Acceleration = f''

Why does $s'(t)$ represent instantaneous velocity?

Let us use t to represent time (in hours) and $s(t)$ the distance (in miles) traveled. So, $s(0.5) = 31$ means that after 30 minutes of traveling you have gone 31 miles and $s(1.2) = 70$ means that after 1.2 hours you have gone 70 miles, etc.



Consider a short interval of time h from $t = 1$ to $t = 1 + h$. During this time, the average speed was

$$\frac{\text{distance}}{\text{time}} = \frac{\text{final position} - \text{initial position}}{\text{final time} - \text{initial time}} = \frac{s(1+h) - s(1)}{h}$$

This is exactly the *difference quotient* we learned before and it approaches the $s'(1)$ as h approaches zero.

Another way to think about this is: if h is small enough, then the velocity at position $s(1+h)$ must be close to the velocity at $s(1)$. Hence, for small enough h, the average velocity $\frac{s(1+h)-s(1)}{h}$ will be close to the instantaneous velocity at $t = 1$. Letting h approach zero, we see that the average velocity $\frac{s(1+h)-s(1)}{h}$ approaches the instantaneous velocity $s'(1)$.

Example: (#4, page 129)
Suppose that

$$f(t) = 3t + 2 - \frac{12}{t}.$$

a) What is the average rate of change of $f(t)$ over the interval from 2 to 3?

Solution: Look at $\dfrac{f(2+h) - f(2)}{h}$ where $h = 3 - 2 = 1$.

$$\begin{aligned}\frac{f(2+1) - f(2)}{1} = \frac{f(3) - f(2)}{1} \\ &= 3 \cdot 3 + 2 - \frac{12}{3} - (3 \cdot 2 + 2 - \frac{12}{2}) \\ &= 9 + 2 - 4 - 6 - 2 + 6 \\ &= 5.\end{aligned}$$

b) What is the (instantaneous) rate of change when $t = 2$?

Solution:

$$\begin{aligned}f'(t) &= 3 + 0 - 12(-1)t^{-2} \\ &= 3 + 12t^{-2}.\end{aligned}$$

Hence,

$$\begin{aligned}f'(2) &= 3 + 12(2)^{-2} \\ &= 3 + \frac{12}{4} \\ &= 6.\end{aligned}$$

Example: (#8, page 129)
After an advertising campaign, the sales of a product often increase and then decrease. Suppose that t days after the end of the advertising, the daily sales are $f(t) = -3t^2 + 32t + 100$ units. What is the average rate of growth in sales during the fourth day, that is, from time $t = 3$ to $t = 4$? At what (instantaneous) rate are the sales changing when $t = 2$?

Solution:

a) Use $\dfrac{f(3+h) - f(3)}{h}$ with $h = 4 - 3 = 1$.

$$\begin{aligned}\frac{f(4) - f(3)}{1} &= -3(4)^2 + 32 \cdot 4 + 100 - (-3(3)^2 + 32 \cdot 3 + 100) \\ &= -3(16 - 9) + 32(4 - 3) \\ &= -21 + 32 \\ &= 11 \ \ units \ \ per \ \ day\end{aligned}$$

b)

$$\begin{aligned}f'(t) &= -3(2)t + 32 + 0 \\ &= -6t + 32\end{aligned}$$

$$\begin{aligned}f'(2) &= -6(2) + 32 \\ &= 20 \ \ units \ \ per \ \ day\end{aligned}$$

Example: (#12, page 130)

Suppose that the position of a car at time t (in hours) is given by $s(t) = 50t - \frac{7}{t+1}$, where the position is measured in kilometers. Find the velocity and acceleration of the car at $t = 0$.

Solution: $s(t) = 50t - \frac{7}{t+1} = 50t - 7(t+1)^{-1}$

$$s'(t) = 50 + 7(t+1)^{-2}$$

$s'(0) = 50 + 7(0+1)^{-2} = 57 \ km \ \ per \ \ hour$ is the velocity at $t = 0$.

$$s''(t) = 7(-2)(t+1)^{-3}$$
$$= -14(t+1)^{-3}$$

$s''(0) = -14 \ \frac{km}{hr^2}$ is the acceleration at $t = 0$.

18. A car is traveling from New York to Boston and is partway between the two cities. Let $s(t)$ be the distance from New York during the next minute. Match each behavior with the corresponding graph of $s(t)$ in Fig. 9.

(a) The car travels at a steady speed.

(b) The car is stopped.

(c) The car is backing up.

(d) The car is accelerating.

(e) The car is decelerating.

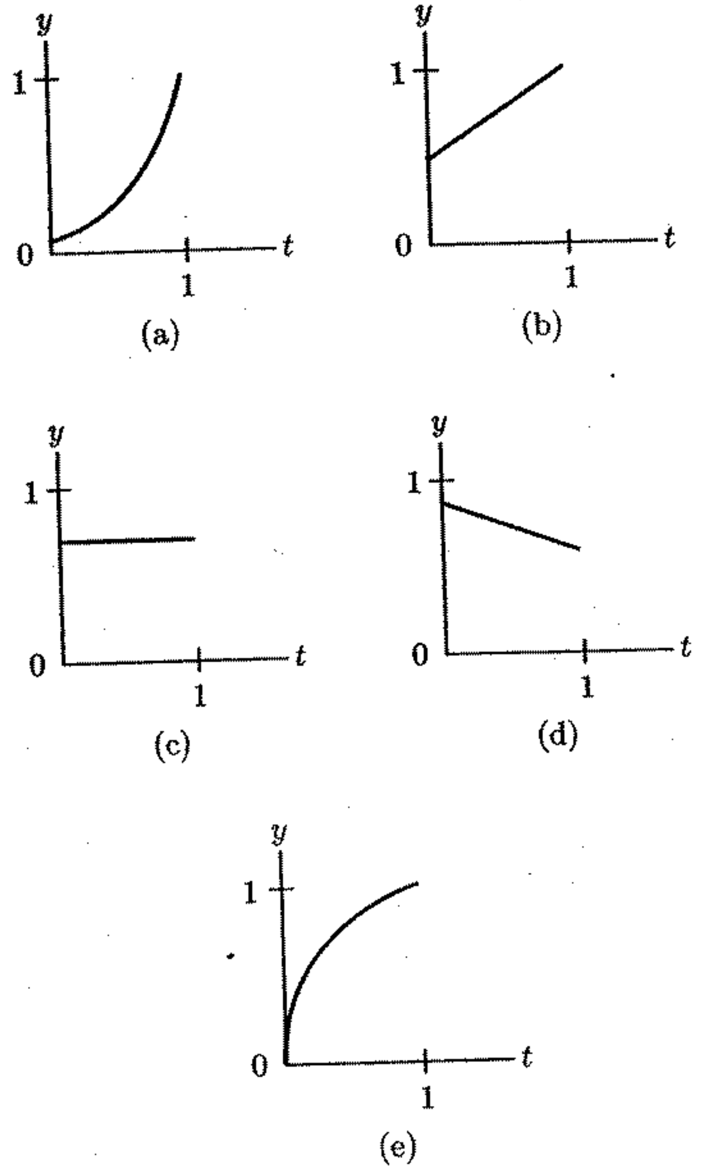


Figure 9. Possible graphs of $s(t)$.

Solution:

List **Picture**

a) -------------→ b) or d) (c)

b) -------------→ c)

c) -------------→ d)

d) -------------→ a)

e) -------------→ e)

Example: (#22, page 131)
Suppose 5 mg of a drug is injected into the bloodstream. Let $f(t)$ be the amount present in the bloodstream after t hours. Interpret $f(3)=2$ and $f'(3)=-.5$. Estimate the number of milligrams of the drug in the bloodstream after $3\frac{1}{2}$ hours.

Solution: (Keyword: Estimate)

i) Interpret $f(3)=2$

After 3 hours there are 2 mg of the drug still in the bloodstream.

ii) Interpret $f'(3)=-.5$

After 3 hours the rate at which the drug is leaving the bloodstream is .5 mg per hour.

iii) *Estimate*

Recall that, for small h, $f'(a)\approx\dfrac{f(a+h)-f(a)}{h}$.

Multiplying by h, $f(a+h)-f(a)\approx f'(a)\cdot h$.

Therefore,

$$f(a+h)\approx f'(a)\cdot h+f(a)$$
$$f(3+.5)\approx f'(3)\cdot .5+f(3)$$
$$f(3.5)\approx(-.5)\cdot .5+2$$
$$\approx -\frac{1}{4}+2$$
$$\approx \frac{7}{4}\ mg\ or\ 1\frac{3}{4}mg$$