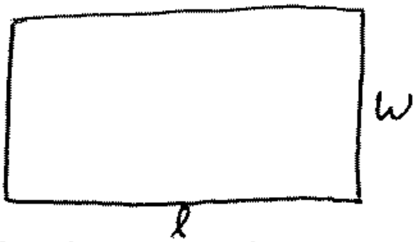
Section 1.5: Max-Min Problems

Goal: Use derivatives to solve problems in which a quantity is to be maximized or minimized.

Example: (#14, page 180) Consider the problem of finding the dimensions of the rectangular garden of area 100 square meters for which the amount of fencing needed to surround the garden is as small as possible.

(a) Draw a picture of a rectangle and select appropriate letters for the dimensions.
(b) Determine the objective and constraint equations.
(c) Find the optimal values for the dimensions.

a)



l = length of rectangle

w = width of rectangle

b) Objective Equation:
Let S = amount of fencing needed.
We want to minimize:

$$S = 2l + 2w$$

Constraint Equation: The area of the fence must be $100\ m^2$. This gives us the constraint:

$$l \cdot w = 100$$

c) constraint: $l \cdot w = 100$, so $l = \frac{100}{w}$.

Substitute the constraint into the objective equation to eliminate one of the variables.

$$S = 2l + 2w = 2(\frac{100}{w}) + 2w = 200 \cdot w^{-1} + 2w.$$

We need to minimize this, so let's sketch it:

Asymptotes: For large w, $200 \cdot w^{-1} + 2w \approx 2w$ ($\lim\limits_{w \to \infty} \frac{200}{w} = 0$), so $y = 2w$ is an asymptote.

For w positive and small, near 0, $200 \cdot w^{-1} + 2w \approx 200 \cdot w^{-1}$; $x = 0$ is an asymptote.

Find critical points: (where is the derivative zero?)

$$\frac{d}{dw}(200 \cdot w^{-1} + 2w) = \frac{-200}{w^2} + 2$$

So $\frac{-200}{w^2} + 2 = 0$ if and only if (clear denominators)

$$-200 + 2w^2 = 0 \qquad w^2 = 100 \qquad w = 10$$

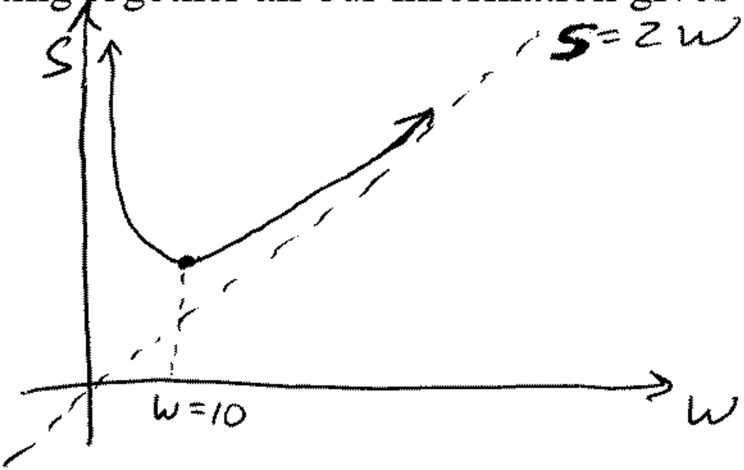
Since $l = 100/w$, $l = 10$.

Second Derivative (concavity)

$$S'' = \frac{d}{dw}S' = \frac{d}{dw}\left(\frac{-200}{w^2} + 2\right) = \frac{400}{w^3}$$

Observe that $S''(10) > 0$, so that S is concave up (holds water) at $x = 10$. In fact, $S'' > 0$ for all $w > 0$, so that C holds water for all $w > 0$.

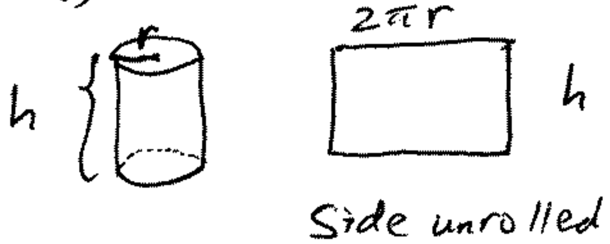
Putting together all our information gives us:



We see that $l = 10$ meters, $w = 10$ meters is a *minimum*.

Example: (#24, page181) A large soup can is to be designed so that the can will hold 16π cubic inches (about 28 ounces) of soup. [See Figure 13(b).] Find the values of r and h for which the amount of metal needed is as small as possible.

Figure 13(b)



h= height

r= radius

Objective Equation: Minimize $S = \pi r^2 \cdot 2 + 2\pi r h$

Constraint Equation: Volume $= V = \pi r^2 \cdot h = 16\pi$

Hence $h = \dfrac{16}{r^2}$.

Substitute constraint into the objective equation:

$$S = \pi r^2 \cdot 2 + 2\pi r(\frac{16}{r^2}) = 2\pi r^2 + 32\pi r^{-1}$$

When r is large, S looks like $2\pi r^2$.

When r is near zero, S looks like $32\pi r^{-1}$.

Critical points: $\frac{d}{dr}S = 4\pi r - 32\pi r^{-2}$

So

$S' = 0$ if and only if $4\pi r - 32\pi r^{-2} = 0$

$$r - 8r^{-2} = 0 \quad \text{(clear denominators)}$$

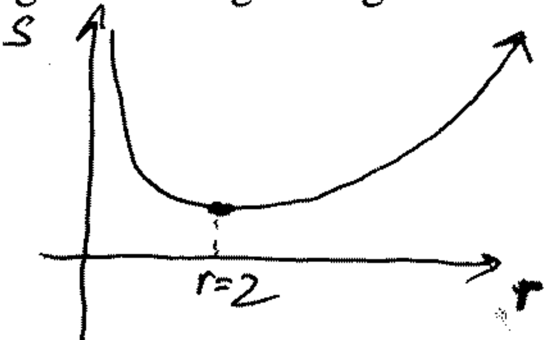
$$r^3 - 8 = 0$$

$$r = 2$$

Since $h = \frac{16}{r^2}$, $h = 4$.

Concavity: $\frac{d^2}{dr^2}S = 4\pi + \frac{64\pi}{r^3}$

Therefore S'' is positive for $r > 0$.
S holds water for $r > 0$.
Putting all this together gives us:



We see that $r = 2$ in., $h = 4$ in. *minimizes* the amount of metal needed.

Example: (#26, page 181) A ship uses $5x^2$ dollars of fuel per hour when traveling at a speed of x miles per hour. The other expenses of operating the ship amount to \$2000 dollars per hour. What speed minimizes the cost of a 500-mile trip? [*Hint*: Express cost in terms of speed and time. The constraint equation is *distance* = *speed* × *time*.]

x = distance in miles $\quad$ t = time in hours

Objective Equation: Minimize

$$C = 5x^2 \cdot t + 2000t$$

Constraint Equation: $500 = x \cdot t$

So $t = \dfrac{500}{x}$.

Substitute constraint into objective equation.

$C = 2500x + 10^6 \cdot x^{-1}$.

When x is large, C looks like $2500x$.

When x is near zero, C looks like $10^6 \cdot x^{-1}$.

Critical points: $\frac{d}{dx}C = 2500 - 10^6 \cdot x^{-2}$

So $C' = 0$ if and only if (clear denominators)

$$2500x^2 - 10^6 = 0 \qquad x^2 - 400 = 0 \qquad x = 20$$

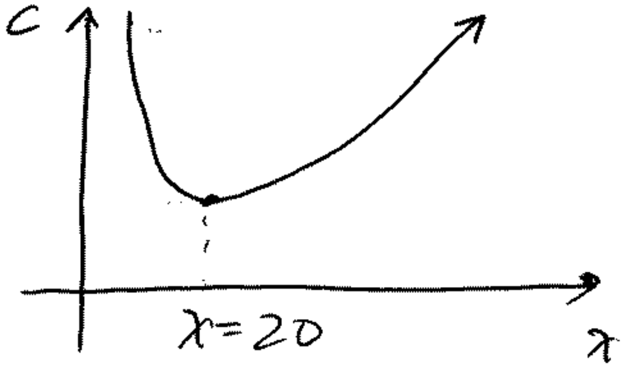
Since $t = \frac{500}{x}$, $t = 50$.

Second Derivative (concavity)

$$\frac{d^2}{dx^2}C = 2 \cdot 10^6 / x^3$$

Hence $C'' > 0$ for $x > 0$
so that C holds water for $x > 0$.

Putting this all together gives us:



We see that $x = 20$ mph gives the minimum cost.