Section 7.1: Functions of Several Variables

Up until now we have been studying functions of the form $y = f(x)$, where the function y depends on the *single* variable x. Of course, in the real world, functions often depend on more than one input. For example, the cost C of manufacturing a single computer might depend on both the cost of labor x, and the cost of the materials involved y. Thus it is important for us to understand functions that depend on several variables.

Definition: A *function of n variables* is a rule that assigns to each n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ a unique real number $f(x_1, x_2, \ldots, x_n)$.

Example:

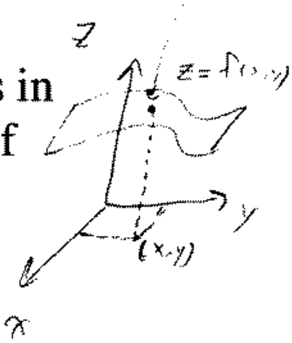
$$f(x, y) = x^2 y^3 + xy + 4y^2$$

is a function of 2 variables. Its value at the point $(x, y) = (2, -1)$ is

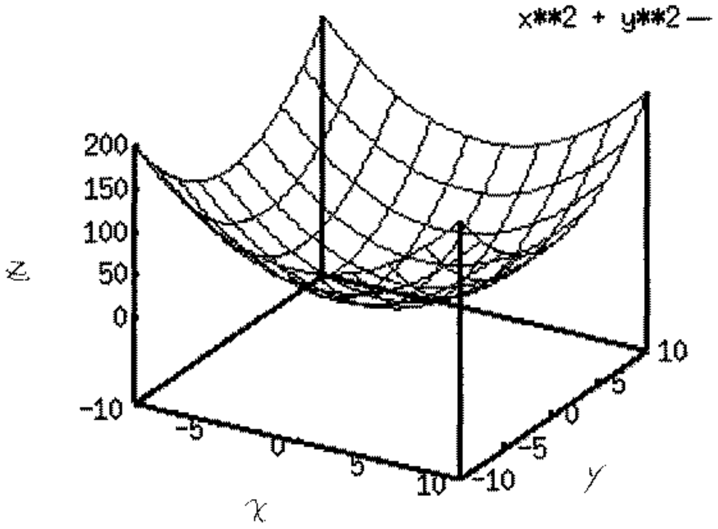
$$f(2,-1) = 2^2 \cdot (-1)^3 + 2 \cdot (-1) + 4 \cdot (-1)^2 = -2.$$

$(x, y, z = f(x,y))$

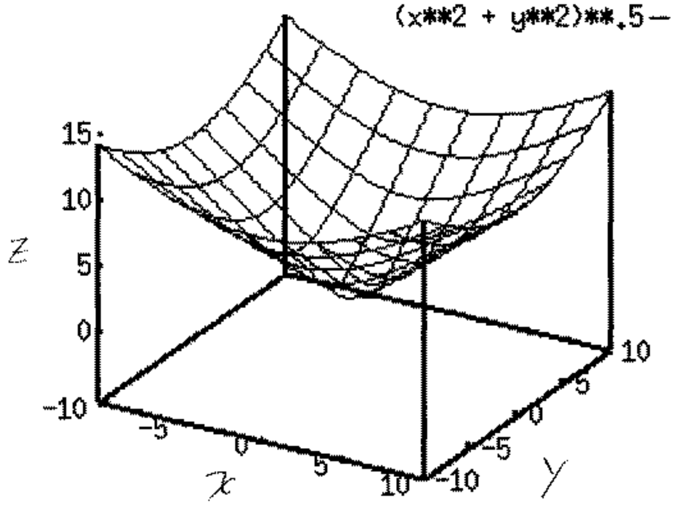
For functions of two variables, their graphs are surfaces in three dimensional space. Some examples of functions of two variables and their graphs are included on the next page.



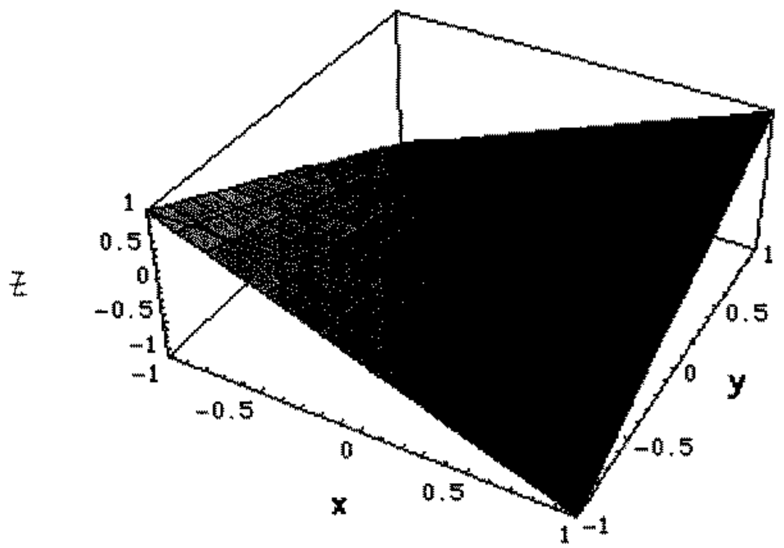
$$z = x^2 + y^2$$



$$z = \sqrt{x^2 + y^2}$$



$$z = xy$$



Example: A company sells two items, sugar and flour. It sells sugar for $4/pound and flour for $5/pound. Find the company's total revenue function and determine its revenue when it sells 100 pounds of sugar and 200 pounds of flour.

Solution: Let

$$x = \text{number of pounds of sugar sold}$$
$$y = \text{number of pounds of flour sold}$$

Then the revenue $R(x, y)$ from selling x pounds of sugar and y pounds of flour is

$$R(x, y) = 4x + 5y.$$

The revenue from selling 100 pounds of sugar and 200 pounds of flour is:

$$R(100, 200) = 4(100) + 5(200) = 1400 \text{ dollars}.$$

Example: (Production Functions in Economics)
The production output of a factory is primarily determined by the amount of labor employed and the capital invested. The function

$$P(L,K) = CL^{\alpha}K^{1-\alpha}$$

where P = production output, L = labor, K = capital, and C is a constant, is called the Cobb-Douglas production function.

a) If

$$P(L,K) = 1.01L^{3/4}K^{1/4},$$

find and interpret $P(147,208)$.

Solution:

$$P(147,208) = 1.01 \cdot 147^{3/4} \cdot 208^{1/4} \approx 161.9$$

When 147 units of labor are employed and 208 units of capital are invested, the production output is approximately 161.9 units.

b) Show that whenever the amounts of labor and capital are doubled, so is the production. (Economists say that the production function has “constant returns to scale.”)

Solution:

Utilization of a units of labor and b units of capital results in a production of $f(a,b)$ units of goods. Utilizing $2a$ units of labor and $2b$ units of capital results in $f(2a,2b)$ units being produced. Now

$$f(2a,2b) = 1.01(2a)^{3/4}(2b)^{1/4}$$

$$= 1.01 \cdot 2^{3/4} \cdot a^{3/4} \cdot 2^{1/4} \cdot b^{3/4}$$

$$= 1.01 \cdot 2^{3/4+1/4} \cdot a^{3/4} \cdot b^{1/4}$$

$$= 2 \cdot 1.01 a^{3/4} b^{1/4}$$

$$= 2f(a,b)$$