<u>Section 7.2</u>: Partial Derivatives

Consider the function of two variables

$$f(x,y) = x^2y^3 + xy + 4y^2$$

Now, we want to differentiate this function. The main idea here is to *treat one of the variables as constant*, and then differentiate with respect of the other variable.

For example, if $y = 3$, then $f(x,3) = 27x^2 + 3x + 36$ is a function of the single variable x. Therefore, we can take the derivative with respect to x.

$$\frac{d}{dx}\left(27x^2 + 3x + 36\right) = 54x + 3$$

We can do the same thing for *any* value of y. This is the idea of the **<u>partial derivative</u>**. In calculus, we write the partial derivative of $f(x,y)$ with respect to x as $\frac{\partial f}{\partial x}$. Here the notation "$\partial f / \partial x$" means we are taking a partial derivative of f, where we treat all variables EXCEPT x as constant. Hence, for $f(x,y) = x^2y^3 + xy + 4y^2$,

$$\frac{\partial f}{\partial x} = 2xy^3 + y.$$

Similarly,

$$\frac{\partial f}{\partial y} = 3x^2y^2 + x + 8y.$$

Example 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for

$$f(x, y) = \frac{x}{1+e^y}.$$

Solution:

To compute $\frac{\partial f}{\partial x}$, treat y as a constant:

$$\frac{\partial f}{\partial x} = \frac{1}{1+e^y}\frac{\partial}{\partial x}x = \frac{1}{1+e^y}$$

To compute $\frac{\partial f}{\partial y}$, treat x as a constant:

$$\frac{\partial f}{\partial y} = x \cdot \frac{\partial}{\partial y}\left(1+e^y\right)^{-1}$$

$$= x \cdot (-1)\left(1+e^y\right)^{-2} \cdot \frac{\partial}{\partial y}\left(1+e^y\right) \quad \text{(chain rule!!!)}$$

$$= x \cdot (-1)\left(1+e^y\right)^{-2} \cdot e^y$$

$$= \frac{-xe^y}{\left(1+e^y\right)^2}$$

Example 2: Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for

$$f(x, y, z) = x^2 - xy + y^2 + 2yz + 2z^2 + z.$$

Solution:

$$\frac{\partial f}{\partial x} = 2x - y + 0 + 0 + 0 + 0 = 2x - y$$

$$\frac{\partial f}{\partial y} = 0 - x + 2y + 2z + 0 + 0 = -x + 2y + 2z$$

$$\frac{\partial f}{\partial z} = 0 - 0 + 0 + 2y + 4z + 1 = 2y + 4z + 1$$

All of these partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \ldots$ are called *first order partial derivatives*. Note that these derivatives are themselves functions of several variables. They have, in fact, the same number of variables as the original function f.

<u>Example 3</u>: Find *all* first order partial derivatives of $f(x, y) = e^{xy} + y\ln(x)$.

Solution:

$$\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) + y \cdot \frac{1}{x} \qquad \text{(chain rule!)}$$

$$= ye^{xy} + \frac{1}{x}$$

$$\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y}(xy) + 1 \cdot \ln(x) \qquad \text{(chain rule!)}$$

$$= xe^{xy} + \ln(x)$$

Example 4: Let $f(x, y) = xye^{2x-y}$. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(x, y) = (1, 2)$.

Solution:

$$\frac{\partial f}{\partial x} = y \cdot \frac{\partial}{\partial x} xe^{2x-y} \qquad \text{(product rule!)}$$

$$= y\left[1 \cdot e^{2x-y} + x \cdot e^{2x-y} \cdot 2\right]$$

$$= e^{2x-y}(y + 2xy)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = e^{2 \cdot 1 - 2}(2 + 2 \cdot 1 \cdot 2) = 6$$

$$\frac{\partial f}{\partial y} = x \cdot \frac{\partial}{\partial x} ye^{2x-y} \qquad \text{(product rule!)}$$

$$= x\left[1 \cdot e^{2x-y} + y \cdot e^{2x-y} \cdot (-1)\right]$$

$$= e^{2x-y}(x - xy)$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = e^{2 \cdot 1 - 2}(1 - 1 \cdot 2) = -1$$

Suppose you already have the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function $f(x, y)$. You can then take the partial derivatives of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. This gives you ***the second order partial derivatives*** *of* $f(x, y)$. The notation and meanings for the second order partial derivatives are:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

In general, we have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

The reasons for this are beyond the scope of this course. We can, however, verify it for a specific function.

Example 5: Find *all* second order partial derivatives of $f(x,y) = x^2y^3 + x^4y + xe^y$.

Solution:

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y + e^y$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 + x^4 + xe^y$$

Hence,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2xy^3 + 4x^3y + e^y\right) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(2xy^3 + 4x^3y + e^y\right) = 6xy^2 + 4x^3 + e^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(3x^2y^2 + x^4 + xe^y\right) = 6xy^2 + 4x^3 + e^y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(3x^2y^2 + x^4 + xe^y\right) = 6x^2y + xe^y$$

Similar to the single variable function situation, we can use partial derivatives to approximate values of functions.

Let $f(x, y)$ be a function of two variables. Then if h and k are small, we have

$$f(a+h,b) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b) \cdot h$$

$$f(a,b+k) \approx f(a,b) + \frac{\partial f}{\partial y}(a,b) \cdot k$$

Example 6: The productivity of a country is given by the Cobb-Douglas function $f(x, y) = 300x^{2/3}y^{1/3}$, where x and y are the amount of labor and capital, respectively.

a) Find the marginal productivities of labor and capital when $x = 125$ and $y = 64$.

Marginal productivity of labor is: $\frac{\partial f}{\partial x}$

Marginal productivity of capital is: $\frac{\partial f}{\partial y}$.

$$\left.\frac{\partial f}{\partial x}\right|_{(125,64)} = \left.300 \cdot \frac{2}{3} x^{-1/3} y^{1/3}\right|_{(125,64)} = \left.200x^{-1/3}y^{1/3}\right|_{(125,64)} = 200 \cdot \frac{1}{5} \cdot 4 = 160$$

$$\left.\frac{\partial f}{\partial y}\right|_{(125,64)} = \left.300 \cdot \frac{1}{3} x^{2/3} y^{-2/3}\right|_{(125,64)} = \left.100x^{2/3}y^{-2/3}\right|_{(125,64)} = 100 \cdot 25 \cdot \frac{1}{16} = \frac{625}{4}$$

b) Use part a) to determine the approximate effect on productivity of increasing capital from 64 to 66 units, while keeping labor fixed at 125 units.

Solution: Use $f(125,64)$ to approximate $f(125,66)$.

$$f(125,66) \approx f(125,64) + \frac{\partial f}{\partial y}(125,64)\cdot 2$$

$$= 300\cdot 25\cdot 4 + \frac{625}{4}\cdot 2$$

$$= 30{,}312.5 \text{ units of output}$$

c) What would be the approximate effect of decreasing labor from 125 units to 124 units while keeping capital fixed at 64 units?

Solution: Use $f(125,64)$ to approximate $f(124,64)$.

$$f(124,64) \approx f(125,64) + \frac{\partial f}{\partial y}(125,64)\cdot(-1)$$

$$= 300\cdot 25\cdot 4 + 160(-1)$$

$$= 20{,}840 \text{ units of output.}$$