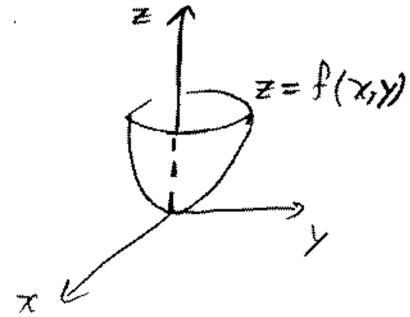
Section 7.3: Relative Minima and Maxima of Functions of Two Variables

Definition: Let f be a function of two variables.
Then

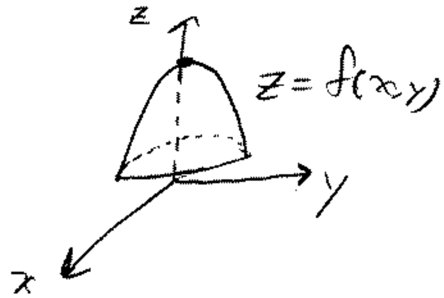


- f has a relative minimum at (a,b) if

$$f(x,y) \geq f(a,b)$$

for all (x,y) near (a,b).

This is just another way to say that $f(a,b)$ is the bottom of a valley on the three-dimensional graph of $f(x,y)$.



- f has a relative maximum at (a,b) if

$$f(x,y) \leq f(a,b)$$

for all (x,y) near (a,b).

This is just another way to say that $f(a,b)$ is the top of a hill on the three-dimensional graph of $f(x,y)$.

How do we find the maximums and minimums on a 3-dimensional graph?

Suppose $f(x,y)$ has a relative minimum at (a,b). Fix $y=b$. Then, as a function of x, $f(x,b)$ has a minimum at $x=a$. Hence,

$$\frac{\partial f}{\partial x}(a,b)=0.$$

Similarly, fixing $x=a$ and looking at $f(a,y)$ as a function of y, we see that

$$\frac{\partial f}{\partial y}(a,b)=0.$$

A similar argument holds for relative maxima, and leads us to...

The First Derivative Test for Functions of Two Variables: If $f(x,y)$ has a relative extreme point at (a,b), then

$$\frac{\partial f}{\partial x}(a,b)=0$$

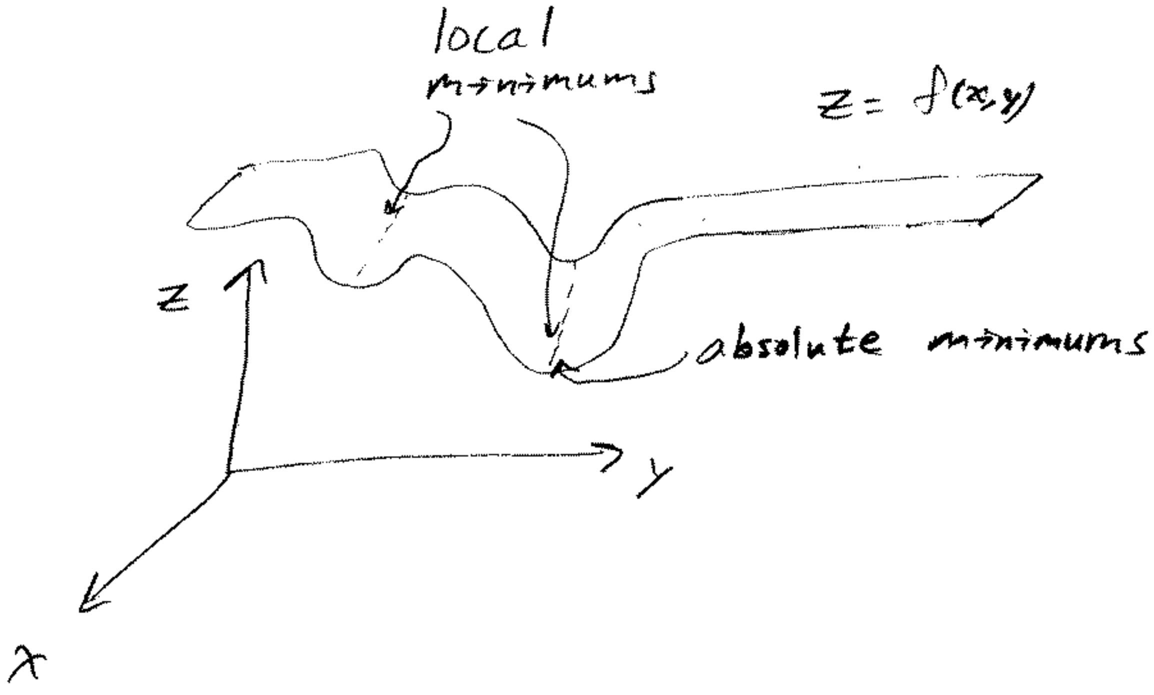
and

$$\frac{\partial f}{\partial y}(a,b)=0.$$

Put another way, this says that the only possible places for maximums and minimums (hills and valleys) to occur on the graph of $f(x, y)$ are those points that simultaneously make both first order partial derivatives equal to zero.

A relative extreme point may be a local extreme or an absolute extreme point, just as for functions of one variable.

Also note that the first derivative test only tells us where ***possible*** relative maxima and minima occur.



Example: Find all possible relative extreme points of

$$f(x,y) = 3x^2 - 4xy + 3y^2 + 8x - 17y + 30.$$

Solution: By the First Derivative Test, we need to find all ordered pairs (a,b) for which both first order partial derivatives are zero; i.e., we need to find all (a,b) satisfying

$$\begin{cases} \dfrac{\partial f}{\partial x} = 6x - 4y + 8 = 0 \\ \dfrac{\partial f}{\partial y} = -4x + 6y - 17 = 0 \end{cases}$$

$2(6x - 4y + 8 = 0)$ $\qquad 12x - 8y + 16 = 0$

$3(-4x + 6y - 17 = 0)$ $\qquad + \; -12x + 18y - 51 = 0$

$$10y - 35 = 0$$

$$y = \frac{35}{10} = 7/2$$

$6x - 4(7/2) + 8 = 0 \Rightarrow 6x - 14 + 8 = 0 \Rightarrow 6x = 6 \Rightarrow x = 1.$

Thus we find that the only possible extreme point of $f(x,y)$ is $(1, 7/2)$.

Example: Find all possible relative extreme points of

$$f(x, y) = \frac{1}{2}x^2 + y^2 - 3x + 2y - 5.$$

Solution: We need to find all points for which

$$\begin{cases} \frac{\partial f}{\partial x} = x - 3 = 0 \\ \frac{\partial f}{\partial y} = 2y + 2 = 0 \end{cases}$$

$x - 3 = 0 \implies x = 3$

$2y + 2 = 0 \implies y = -1$

Hence, $(3, -1)$ is the only possible extreme point.

O.K. – we can find *possible* maximums and minimums, but how do we tell if we really have a maximum or really have a minimum?

Excellent question! Just as for functions of one variable, we have a *second derivative test*! ☺

<u>The Second Derivative Test</u>: Suppose $f(x,y)$ is a function of two variables, and suppose that the point (a,b) satisfies

$$\begin{cases} \dfrac{\partial f}{\partial x}(a,b) = 0 \\ \dfrac{\partial f}{\partial y}(a,b) = 0 \end{cases}$$

Set

$$D(x,y) = \left(\frac{\partial f^2}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

Then

1) If $D(a,b) > 0$

   a) and $\dfrac{\partial^2 f}{\partial x^2}(a,b) > 0$, then the point (a,b) is a relative minimum for $f(x,y)$.

   b) and $\dfrac{\partial^2 f}{\partial x^2}(a,b) < 0$, then the point (a,b) is a relative maximum for $f(x,y)$.

2) If $D(a,b) < 0$, then the point (a,b) is a <u>saddle point</u>; that is, the point (a,b) is neither a maximum nor a minimum.

3) If $D(a,b) = 0$, then the second derivative test fails – it gives us no information about whether (a,b) is a relative minimum, maximum, or neither.

Example: Find and identify the relative extreme points of

$$f(x, y) = y^3 - x^2 + 6x - 12y + 5.$$

Solution: First, find all possible extreme points.
Solve:

$$\begin{cases} \dfrac{\partial f}{\partial x} = -2x + 6 = 0 \\ \dfrac{\partial f}{\partial y} = 3y^2 - 12 = 0 \end{cases}$$

$3y^2 - 12 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

$-2x + 6 = 0 \Rightarrow x = 3$

Thus, $(3, 2)$ and $(3, -2)$ are the only possible extreme points for $f(x, y)$.

Next, we apply the second derivative test to our points. To construct $D(x, y)$, we find

$$\frac{\partial^2 f}{\partial x^2} = -2 \qquad \frac{\partial^2 f}{\partial y^2} = 6y \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

Hence,

$$D(x, y) = \left(\frac{\partial f^2}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (-2)(6y) - (0)^2 = -12y.$$

Thus,

- $D(3,2) = -12(2) = -24 < 0$.

Therefore, $(3,2)$ is a saddle point (neither a maximum nor a minimum) by the second derivative test.

- $D(3,-2) = -12(-2) = 24 > 0$.

Also, $\frac{\partial^2 f}{\partial x^2}(3,-2) = -2 < 0$.

Therefore, $(3,-2)$ is a local maximum for f by the second derivative test.

Example: A firm makes two kinds of golf balls; one sells for \$3 and the other for \$2. If x is the number of the first type produced and y is the number of the second type, then the revenue function is given by

$$R(x,y) = 3x + 2y$$

and the cost function is given by

$$C(x,y) = 2x^2 - 2xy + y^2 - 9x + 6y + 7 .$$

Find the level of production (a,b) that maximizes the company's *profit*.

Solution: The profit function is obtained from the revenue and cost functions as follows:

$$P(x,y) = R(x,y) - C(x,y)$$

$$= -2x^2 + 2xy - y^2 + 12x - 4y - 7.$$

Then

$$\begin{cases} \frac{\partial P}{\partial x} = -4x + 2y + 12 = 0 \\ \frac{\partial P}{\partial y} = 2x - 2y - 4 = 0 \end{cases}$$

$-4x + 2y + 12 = 0$

$2(2x - 2y - 4 = 0)$

$-4x + 2y + 12 = 0$

$+4x - 4y - 8 = 0$

10

$-2y + 4 = 0$

$y = 2$

$-4x + 2(2) + 12 = 0$

$-4x = -16$

$x = 4$

Hence $(4,2)$ is the only possible extreme point.
Now apply the second derivative test to determine if the point is a relative maximum, minimum, or neither.

$$\frac{\partial^2 P}{\partial x^2} = -4 \qquad \frac{\partial^2 P}{\partial y^2} = -2 \qquad \frac{\partial^2 P}{\partial x \partial y} = 2$$

Hence

$$D(x, y) = \left(\frac{\partial f^2}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (-4)(-2) - (2)^2 = 4.$$

Therefore,

$$D(4,2) = 4 > 0$$

Since also

$$\frac{\partial^2 P}{\partial x^2} = -4 < 0,$$

the point $(4,2)$ is a relative maximum for $P(x, y)$ by the second derivative test.