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Math 220-X8 **Test 3** **Name** Solutions

10 ☐ **Calculate the Indefinite Integral [2 pt each].**

1) $\int 7\, e^x\, dx$

$= 7e^x + C$

2) $\int x^{-3}\, dx$

$= \frac{x^{-2}}{-2} + C$

3) $\int (3 - x)\, dx$

$= 3x - \frac{x^2}{2} + C$

4) $\int \left(\text{Cos}[x] + \frac{1}{x}\right) dx$

$= \text{Sin}(x) + \text{Log}|x| + C$

5) $\int \frac{\text{Cos}[e^x]e^x}{\text{Sin}[e^x]}\, dx$

$= \text{Log}\,|\text{Sin}(e^x)| + C$

15 ☐ **Calculate the Definite Integral [3 pts each]. Show some steps.**

6) $\int_0^1 e^x\, dx$

$= e^x\Big|_0^1 = e^1 - e^0$

$= e - 1.$

7) $\int_1^2 (x^{-2} + 3x)\, dx$

$= \frac{x^{-1}}{-1} + 3\frac{x^2}{2}\Big|_1^2$

$= \left(-\frac{1}{2} + \frac{1}{1}\right) + \frac{3}{2}(2^2 - 1^2) = \frac{1}{2} + \frac{9}{2}$

$= 5.$

8) $\int_0^2 \left(\frac{1}{x^4} - 3x + 1\right) dx$

$= \frac{x^{-3}}{-3} - \frac{3}{2}x^2 + x\Big|_0^2$

$= -\frac{1}{3}\left(\frac{1}{2^3} - \frac{1}{0^3}\right) - \frac{3}{2}(2^2 - 0) + (2 - 0) = \left(\text{D.N.E. or } \infty\right)$

undefined!

9) $\int_0^{\pi/4} \frac{\text{Sin}[x]}{\text{Cos}[x]}\, dx$

$= -\Big[\text{Log}\,|\text{Cos}(x)|\Big]_0^{\pi/4}$

$= -\left[\text{Log}\left(\frac{\sqrt{2}}{2}\right) - \text{Log}(1)\right]$

$= -\text{Log}\left(\frac{\sqrt{2}}{2}\right) = -\text{Log}\left(2^{-1/2}\right)$

$= \frac{1}{2}\text{Log}(2).$

10) $\int_0^2 (x^3 + 3x - 1)\, dx$

$= \frac{x^4}{4} + \frac{3}{2}x^2 - x\Big]_0^2$

$= \frac{1}{4}(2^4 - 0) + \frac{3}{2}(2^2 - 0) - (2 - 0) = 4 + 6 - 2$

$= 8.$

☐ **True False False [2 pts each].** ***If False, you must explain why.***

11) If f[x] is a function with even symmetry, then $\int_{-2}^{2} f[x]\, dx = 0$.

False. If f[x] is odd then that integral is zero.

12) If f[x] is a function with odd symmetry, then $\int_{-6}^{6} f[x]\,dx = 0$.

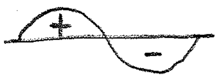
True.

13) If f[x] is a function with odd symmetry, then $\int_{-1}^{1} f[x]\,dx = 2\int_{0}^{1} f[x]\,dx$.

False. This happens when $f[x]$ is <u>even</u>.

14) $\int_{0}^{\pi} \text{Sin}[x]\,dx = -\int_{\pi}^{2\pi} \text{Sin}[x]\,dx$.

True.

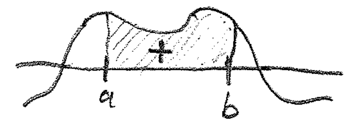


15) $\int_{a}^{b} f[x]\,dx$ is a number, but $\int f[x]\,dx$ is a function.

True.

16) If f[x] > 0 for all x in the interval [a, b], then we know that $\int_{a}^{b} f[x]\,dx > 0$.

True.



17) $\int_{0}^{a} \frac{1}{x}\,dx = \text{Log}[a] - \text{Log}[0]$.

False. Log[0] is undefined. $\frac{1}{x}$ has a vertical asymptote at $x = 0$.

18) $\int_{a}^{b} x^3\,dx = \frac{x^4}{4} + C$.

False. The left side is definite, right side indefinite.

19) $\int_{a}^{b} (f[x] + g[x])\,dx = \int_{a}^{b} f[x]\,dx + \int_{a}^{b} g[x]\,dx$.

True.

20) $\int_{a}^{b} f[x]\,g[x]\,dx = \left(\int_{a}^{b} f[x]\,dx\right)\left(\int_{a}^{b} g[x]\,dx\right)$.

False. There is no product rule for integrals.
Counterexample: $\int 3\cdot 2\,dx = 6x \neq 6x^2 = \left(\int 3\,dx\right)\left(\int 2\,dx\right)$

21) $\left|\int_{-1}^{1} x\,dx\right| = \int_{-1}^{1} |x|\,dx$.

False.

$\left|\int_{-1}^{1} x\,dx\right| = \left|\left[\frac{x^2}{2}\right]_{-1}^{1}\right| = \left|\frac{1}{2} - \frac{1}{2}\right| = 0.$

$\int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} x\,dx = 2\left.\frac{x^2}{2}\right]_{0}^{1} = 1.$

☐ Multiple Choice [3 pts each]. Choose the letter of the best answer.

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22) Consider the function y[x] which satisfies the differential equation $y'[x] = y[x]\left(2 + x^2 - y[x]\right)$ and $y[x] > 0$ for all x. We know that y[x] will be decreasing during those intervals which make __A__ true.

A) $y[x] > 2 + x^2$ B) $y[x] > x^2$ C) $y[x] < 0$ D) $y[x] < 2 + x^2$

$y'[x] < 0$ when
$2 + x^2 - y[x] < 0$
$2 + x^2 < y[x]$

−2 for "D"

23) Consider the same function y[x] as the previous problem. If y[x] has a minimum at x = a, then we know that __A__ must be true.

A) $y[a] = 2 + a^2$ B) $y[a] = 0$ C) $y[a] = 0$ or $y[a] = 2 + a^2$ $y'(x) = 0$

D) $2 + a^2 = 0$ $y(x) > 0$

(-1.5 for "C")

24) The usual predator-prey model involves the simultaneous differential equations

prey′[t] = a prey[t] − b prey[t] pred[t] , $= a\,\text{prey}(t) - b\,\text{prey}(t)\left[\frac{a}{b} - 1\right] = +b\,\text{prey}(t) > 0$

pred′[t] = −c pred[t] + d pred[t] prey[t] , $= -c\,\text{pred}(t) + d\,\text{pred}(t)\left[\frac{c}{d} - 1\right] = -d\,\text{pred}(t) < 0$

where a, b, c, and d are given positive constants. Which one of these statements is true? __B__

A) When prey[t] < $\frac{c}{d}$, pred[t] is going up. B) When pred[t] < $\frac{a}{b}$, prey[t] is going up.

C) When prey[t] = $\frac{c}{d}$, pred[t] has a max. or min! D) When pred[t] = 0 , prey[t] is going down.

(-2 for "C") ↳ makes prey′(t) > 0

25) Using the same predator-prey model as above, which of these statements is true? __C__

A) If pred[t] = 0, then prey[t] won't change. B) If pred[t] = prey[t], then t = 0 .

C) If pred[t] = $\frac{a}{b}$, then prey[t] is neither increasing nor decreasing. ↔ prey′(t) = 0

D) If a = b, then prey[t] is neither increasing nor decreasing. ↔ a prey − a prey pred = a prey (1 − pred)

26) The SIR infection model dealing with disease in a closed population (total population is constant) uses the three functions Sus[t], Inf[t], and Recov[t], which satisfy these simultaneous differential equations (a and b are positive constants):

Sus′[t] = −a Sus[t] Inf[t],

Inf′[t] = a Sus[t] Inf[t] − b Inf[t], $a\left(\frac{b}{a} + 1\right)\text{Inf}(t) - b\,\text{Inf}(t) = a\,\text{Inf}(t) \geq 0$

Recov′[t] = b Inf[t] .

After someone becomes infected and recovers from this disease, they become immune. Which of the following statements is FALSE? __D__

A) If Sus[0] = 0, then there will be no epidemic. ✓ B) If Inf[0] = 0, then there will be no epidemic. ✓

C) If Inf[t] and Sus[t] are positive, then Sus[t] is decreasing. ✓

D) If Sus[t] > $\frac{b}{a}$, then Inf[t] is decreasing.

27) Using the same predator-prey model as above, which of these statements is FALSE? __C__

A) As Inf[t] increases, Sus[t] decreases. ✓ B) As Inf[t] decreases, Recov[t] increases. ✓

C) If Sus[t] = $\frac{b}{a}$, then Sus[t] has its maximum. ✗ D) If Sus[t] = $\frac{b}{a}$, then Inf[t] has its maximum. ✓

$\text{Sus}'(t) \neq 0$ $\text{Inf}'(t) = 0$

28) If $f[t] = \int_0^t 4\, e^x \,\text{Sin}[x^3]\, dx$, then f′[t] = __A__ .

A) $4\, e^t \,\text{Sin}[t^3]$ B) $4\, e^t \,\text{Cos}[t^3]\,(3\, t^2)$ C) $4\, e^t \int_0^t \text{Sin}[x^3]\, dx$ D) $\int_0^t 4\, e^x \,\text{Cos}[x^3]\,(3\, x^2)\, dx$

29) If $f[t] = \int_0^t \text{Sin}[x]\, dx$, then $f\left[\frac{\pi}{2}\right]$ equals __B__ .

A) The area under −Cos[x] from 0 to $\frac{\pi}{2}$. B) The area under Sin[x] from 0 to $\frac{\pi}{2}$.

C) $\text{Sin}\left[\frac{\pi}{2}\right]$ D) $\text{Tan}\left[\frac{\pi}{2}\right]$

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Various [2 pts each]. As always, show your work.

30) Look at the differential equation $y'[x] = y[x]\,(4 + \text{Sin}[x]^2 - y[x])$ with $y[0] = 7$.
How do you know before you do any plotting that as x advances from 0, the plot of the solution y[x] initially must go down?

Because $y'(0) = y(0)(4 + \sin(0)^2 - y(0))$
$= 7(4 + 0 - 7) < 0$.

31) The acceleration of a falling object is $-32\ \frac{\text{ft}}{\text{s}^2}$. How far has a raindrop fallen 10 seconds after it leaves the cloud?

$v(t) = \int -32\,dt = -32t + C$

$v(0) = 0 + C = 0$
$C = 0$

$d(t) = \int v(t)\,dt = -32\frac{t^2}{2} = -16t^2 + k$

$d(0) = 0 + k = 0$
$k = 0$

$d(10) = -16(10)^2 = -1600$ feet.

32) You have a box attached to a spring which is anchored to the ceiling. You hold the box up to the ceiling and let it go; it will fall down then spring back up toward the ceiling. The acceleration of the box is modeled by $a[t] = -2\,\text{Cos}[t]$. How much time has passed when the box arrives back up at the ceiling? (Note: ~~velocity~~ distance = 0 when this happens.)

$v(t) = \int -2\cos(t)\,dt = -2\sin(t) + C \Leftrightarrow 0 = v(0) = 0 + C$
$\underline{0 = C}$

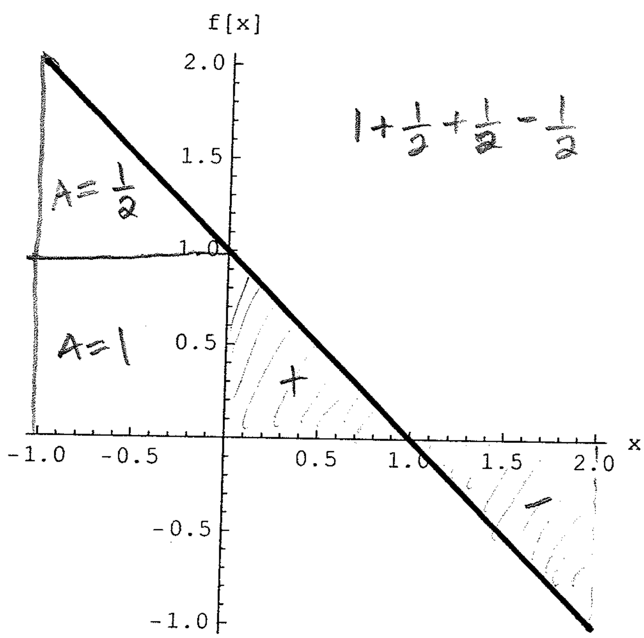
$d(t) = \int -2\sin(t)\,dt = 2\cos(t) + k$

$0 = d(0) = 2(1) + k$
$\underline{-2 = k}$

$d(t) = 2\cos(t) - 2 = 0$
$\cos(t) = 1$
$t = \boxed{2\pi \text{ Seconds}}$

33) Below is the graph of a function f[x] over the interval [−1, 2]. Use any method to calculate $\int_{-1}^{2} f[x]\,dx$.

$= \sum \text{Areas} = 1.5$



$1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}$

or

$f(x) = 1 - x$

$\int_{-1}^{2} (1-x)\,dx$

$= \left[x - \frac{x^2}{2}\right]_{-1}^{2}$

$= (2 - -1) - \frac{1}{2}(2^2 - (-1)^2)$

$= 3 - \frac{1}{2}(3) = 1.5$

34) Explain why sometimes Neptune is farther away from the Sun than Pluto is.

The eccentricity of Pluto's elliptical orbit is much larger than Neptune's eccentricity. The orbits look like

35) The logistic predator-prey model has the simultaneous differential equations

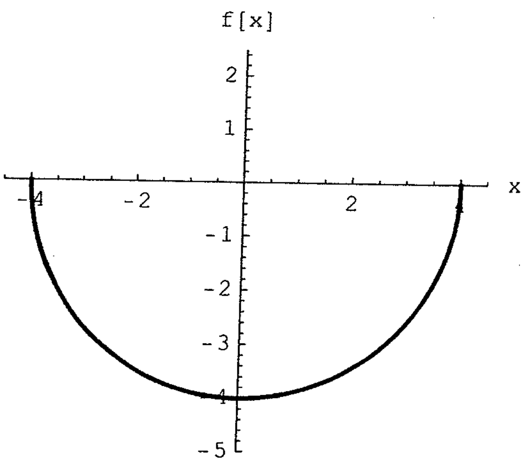
$$\text{prey}'[t] = a\,\text{prey}[t]\left(1 - \frac{\text{prey}[t]}{v}\right) - b\,\text{prey}[t]\,\text{pred}[t],$$

$$\text{pred}'[t] = -c\,\text{pred}[t] + d\,\text{pred}[t]\,\text{prey}[t]\,.$$

Explain why this model is more realistic than the simple predator-prey model (shown in question #24).

This one uses logistic growth for the reproduction of prey. "v" is the maximum prey population its environment can sustain. The other model has unlimited prey.

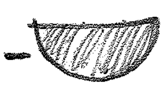
36-37) When you plot $f[x] = -\sqrt{16 - x^2}$ for $-4 \le x \le 4$, you get the bottom of the circle of radius 4 centered at {0, 0}:



The entire circle would have area

$$A = \pi r^2$$
$$= 16\pi.$$

~~That's the bottom half of a circle of radius 2.~~ Use this fact to calculate these two integrals:

36) [2 pts] $\int_{-4}^{4} f[x]\,dx$ $=$  $= -\frac{1}{2}\,16\pi = -8\pi.$

37) [2 pts] $\int_{0}^{4} f[x]\,dx.$ $=$  $= -\frac{1}{4}\,16\pi = -4\pi.$

Below x-axis
$\rightarrow$ negative

☐ Matching [2 pts each]. 12

Match the parametrization with its plot. Write the correct Letter in the blank.

38) C $x[t] = \text{Cos}[t], \quad y[t] = 3\,\text{Sin}[t], \quad 0 \le t \le \pi$

39) D $x[t] = 2t - 1, \quad y[t] = t, \quad 0 \le t \le 1$

40) A $x[t] = 2\,\text{Cos}[t], \quad y[t] = 2\,\text{Sin}[t], \quad 0 \le t \le \frac{\pi}{2}$

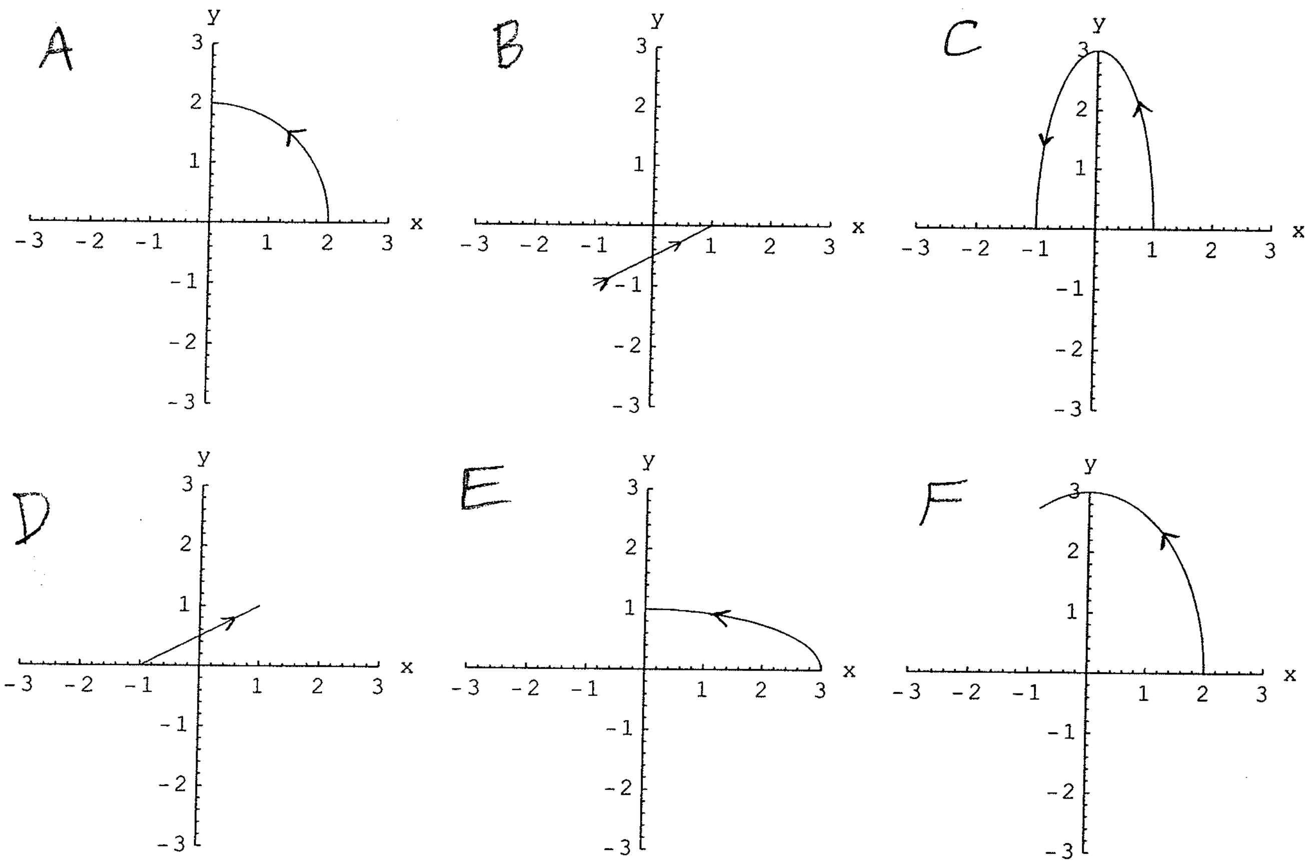
41) E $x[t] = 3\,\text{Cos}[t], \quad y[t] = \text{Sin}[t], \quad 0 \le t \le \frac{\pi}{2}$

42) F $x[t] = 2\,\text{Cos}[2t], \quad y[t] = 3\,\text{Sin}[2t], \quad 0 \le t \le 1$

43) B $x[t] = 2t - 1, \quad y[t] = t - 1, \quad 0 \le t \le 1$

E ↔ C
-3

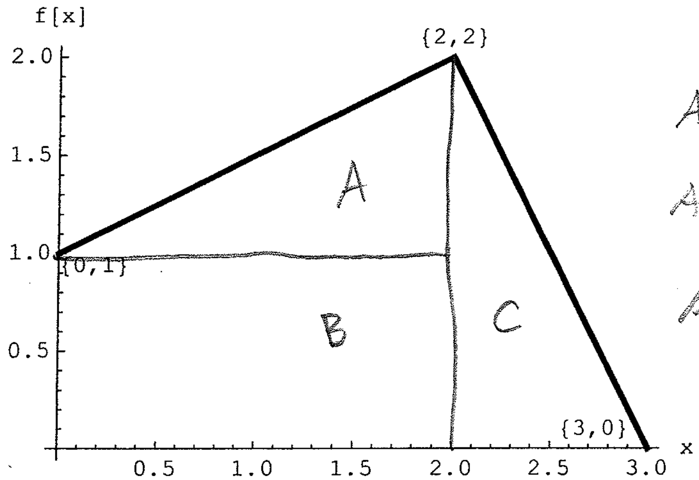
B ↔ D
-2



+5

Bonus

B1) [2 pts] The graph of a function f[x] consists of two straight line segements. The first segment connects {0, 1} and {2, 2} and the second segment connects {2, 2} and {3, 0}. Here's the plot:



Area (A) $= \frac{1}{2}\cdot 2 \cdot 1 = 1$

Area (B) $= 2\cdot 1 = 2$

Area (C) $= \frac{1}{2}\cdot 1 \cdot 2 = 1$

Measure some areas to come up with a calculation of

$\int_0^3 f[x]\, dx.$ $= 1+2+1 = 4$

B2) [3 pts]

- Calculate $\frac{d}{dx}(x \operatorname{Log}[x] - x)$. $= 1 \operatorname{Log}(x) + x\cdot\frac{1}{x} - 1 = \operatorname{Log}(x).$

- Use that information to calculate $\int_1^{e^5} \operatorname{Log}[x]\, dx.$

$$= \left[x \operatorname{Log}(x) - x\right]_1^{e^5}$$

$$= \left[e^5 \operatorname{Log} e^5 - e^5\right] - \left[1 \operatorname{Log}(1) - 1\right]$$

$$= 5e^5 - e^5 - [0-1]$$

$$= 4e^5 + 1.$$