RESEARCH STATEMENT

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My primary area of interest, motivated by results which are both aesthetically pleasing and widely applicable, is convex and discrete geometry. Among the topics in convex and discrete geometry, I am particularly interested in triangulations of lattice polytopes and the geometry of the lattice points contained within these polytopes. My current research focuses on relationships between the volume of a "clean" simplex and the number k of interior lattice points, and the implications thereof. I am also interested in methods for lattice point enumeration, and the connections between convex and discrete geometry and other areas such as combinatorics and number theory. For examples of applications in combinatorics, algebraic number theory, commutative algebra, optimization, linear programming, and statistical physics, I respectfully defer to a recent paper by J. De Loera [4] and a slightly older paper by J.–M. Kantor [12].

1. Background

My interest in convex and discrete geometry developed while assisting my adviser Bruce Reznick with calculations involving "clean" lattice tetrahedra [27]. In general, a *lattice* polytope has vertices with integer coordinates. A *clean* lattice tetrahedron T has the property that its vertices are the only lattice points on its boundary ∂T. If there are k lattice points in the interior $\text{int}(T)$, then T is a *k-point* lattice tetrahedron. A *unimodular transformation* is a map $f : \mathbb{Z}^d \to \mathbb{Z}^d$ defined by $f(v) = v \cdot \mathbf{M} + u$, where $u \in \mathbb{Z}^d$, $\mathbf{M} \in GL(d, \mathbb{Z})$, and $\det(\mathbf{M}) = \pm 1$. Two lattice tetrahedra are *unimodularly equivalent* if one is the image of the other under some unimodular map. Reznick proved in [26] and [27] that all clean k-point tetrahedra are unimodularly equivalent to some $T_{a,b,n}$, the tetrahedron with vertex set

$$\{\ (0,0,0),\ (1,0,0),\ (0,1,0),\ (a,b,n)\ \},$$

where $(a, b, n) \in \mathbb{Z}^3$ and $0 < a \le b < n$. Our initial calculations indicated that for $k \ge 1$, $3k + 1 \le n \le 12k + 8$, or equivalently for any clean tetrahedron T,

$$3k + 1 \le 6 \cdot \text{Vol}(T) \le 12k + 8. \tag{1}$$

C. Bey, M. Henk, and J. Wills [2] recently proved a generalized form of the lower bound in (1); the upper bound in (1) remains unproved. Interestingly, there exist clean polytopes with no lattice interior points and arbitrarily large volume. For $k \ge 1$, however, it is known that the volume of a lattice polytope P of dimension d (or a *d-polytope*) can be bounded from above in terms of the number k and d, as shown by D. Hensley [11]. In 1991, J. Lagarias and G. Ziegler improved upon Hensley's results by showing that

$$\text{Vol}(P) \le k \cdot [7(k + 1)]^{d \cdot 2^{d+1}}. \tag{2}$$

The bound was further improved by O. Pikhurko to

$$\mathrm{Vol}(P) \le (8d)^d \cdot 15^{d \cdot 2^{2d+1}} \cdot k. \tag{3}$$

This appears to be the best upper bound to date for $d \ge 3$. To what extent can this upper bound be improved? In dimension 2, the answer is given by a theorem of G.A. Pick [21].

1.1 Pick's Theorem. *The area A of a polygon P with vertices in $\mathbb{Z}^2$, k interior lattice points, and b lattice points on its boundary is given by*

$$A = k + \frac{b}{2} - 1. \tag{4}$$

For clean polygons, (4) reduces to $A = \frac{1}{2}(2k+1)$. Pick's theorem naturally raises the question: does (4) have analogues in higher dimensions? J. Reeve studied this question in the 1950s (cf. [24], [25]) and concluded that Pick's theorem does not generalize to higher dimensions without extra considerations. Let L_n^d denote the lattice consisting of points in $\mathbb{R}^d$ whose coordinates are multiples of $\frac{1}{n}$. For a given polytope P, let $k_n = |\,\mathrm{int}(P) \cap L_n^d|$ and $b_n = |\partial P \cap L_n^d|$. Reeve discovered that if $d = 3$, then

$$2n(n^2-1) \cdot \mathrm{Vol}(P) = b_n - nb_1 + 2(k_n - nk_n) + (n-1)[2\chi(P) - \chi(\partial P)], \tag{5}$$

where $n \ge 2$ and χ is the Euler characteristic. In 1963, MacDonald generalized Reeve's results and showed that

$$\begin{aligned}(d-1)d! \cdot \mathrm{Vol}(P) = \sum_{i=1}^{d-1} (-1)^{i-1} \binom{d-1}{i-1} (b_{d-i} + 2k_{d-i}) \\ +(-1)^{d-1}[2\chi(P) - \chi(\partial P)].\end{aligned} \tag{6}$$

Then in 1996, Kolodziejczyk [14] showed that (6) has an alternate form which uses only the numbers k_n. He also showed that for polytopes whose vertices have exactly d edges, the new formula reduces to

$$d! \cdot \mathrm{Vol}(P) = \sum_{i=0}^{d-1} (-1)^i \binom{d}{i} k_{d-i} + 1. \tag{7}$$

Computing $k_n = |P \cap L_n^d|$ is equivalent to counting the lattice points in $nP \cap \mathbb{Z}^d$, where nP denotes the nth dilation of P. This suggests that Erhart theory [7] may be useful in improving the upper bound of the volume of Vol(P) in terms of $k = |P \cap \mathbb{Z}^d|$. However, it also possible to use triangulations to determine geometric and combinatorial properties of k-point lattice simplices. In particular, the lower bound in (1) follows almost immediately from properties of triangulations.

2. Triangulations and Refinements

The properties and combinatorics of triangulations depends on the precise notion of the term "triangulation." I use C.L. Lawson's [17] definition of a triangulation. Let $\mathscr{P}$ denote a set of n distinct points in $\mathbb{R}^d$ not lying entirely in a hyperplane, where $n \ge d+1$ and $d \ge 2$. A *triangulation*, $\mathscr{T}$, of $\mathscr{P}$ (or of $P = \mathrm{conv}(\mathscr{P})$ with the dependence on $\mathscr{P}$ understood) is a set of nondegenerate d-simplices $\{T_i\}$ with the following properties.

(a) All vertices of each simplex are members of $\mathscr{P}$.
(b) The interiors of the simplices are pairwise disjoint.

(c) Each facet of a simplex is either on the boundary of P, or else is a common facet of exactly two simplices.
(d) Each simplex contains no points of $\mathscr{P}$ other than its vertices.
(e) The union of $\{T_i\}$ is $\mathscr{P}$ and the union of T_i is P.

For a given lattice d-polytope P, a *lattice triangulation* $\mathscr{T}$ of P is a triangulation of some $\mathscr{P} \subseteq P \cap \mathbb{Z}^d$. The dependency set $\mathscr{P}$, or *vertex set* of $\mathscr{T}$, is simply the set of all vertices of all simplices in $\mathscr{T}$. Using $\mathscr{V}$ to denote vertex set, the points in $\mathscr{V}(P)$ are the vertices of P, and $\mathscr{V}(\mathscr{T}) = \mathscr{P}$. Note P may have several lattice triangulations whose vertex sets are of different cardinality. Given two lattice triangulations $\mathscr{T}$ and $\mathscr{T}'$ of P, we say $\mathscr{T}'$ is a *refinement* of $\mathscr{T}$ (and write $\mathscr{T} \prec \mathscr{T}'$) provided $\mathscr{T}'$ is a lattice triangulation of P, $\mathscr{V}(\mathscr{T}) \subsetneq \mathscr{V}(\mathscr{T}')$, and for all $T' \in \mathscr{T}'$ there exists $T \in \mathscr{T}$ such that $T' \subseteq T$. We say that $\mathscr{T}$ is a *full lattice triangulation* if $\mathscr{V}(\mathscr{T}) = P \cap \mathbb{Z}^d$. Otherwise we say $\mathscr{T}$ is a *partial lattice triangulation*.

2.1 Theorem. [1, Theorem 3.1] *Every convex polytope P can be triangulated using no new vertices. That is, there exists a triangulation of $\mathscr{V}(P)$.*

If P is clean and has an interior lattice point w, we can construct a *basic triangulation* $\mathscr{T}_w$ of P with respect to w as follows. First, triangulate each facet of P as a $(d-1)$-polytope using Theorem 2.1 and let $\mathscr{F}$ be the union of all such triangulations. Then take $\mathscr{T}_w$ to be

$$\mathscr{T}_w = \bigcup_{S \in \mathscr{F}} \{ \operatorname{conv}(S, w) \}.$$

It is easy to check that $\mathscr{T}_w$ is indeed a triangulation and that $|\mathscr{T}_w| \geq d+1$ with equality achieved when P is a clean simplex. Partial triangulations and refinements naturally lead to the notion of sequences of triangulations, or *refinement sequences*, that begin with basic triangulations.

3. Results from Dissertation Research

For this section, P will always denote a nondegenerate lattice d-polytope with $k \geq 1$ interior lattice points. The following result appears (in a slightly different form) in the proof of the inequality

$$\operatorname{Vol}(P) \geq \frac{1}{d!}(dk+1) \tag{8}$$

by Bey, Henk, and Wills [2]. I have adapted it for my dissertation as follows.

3.1 Theorem. *Suppose $\mathscr{T}$ is a partial triangulation of P. For any $w \in \operatorname{int}(P) \cap \mathbb{Z}^d \backslash \mathscr{V}(\mathscr{T})$ there exists an integer $j > 0$ and a refinement $\mathscr{T}'$ of $\mathscr{T}$ such that $\mathscr{V}(\mathscr{T}') = \mathscr{V}(\mathscr{T}) \cup \{w\}$ and $|\mathscr{T}'| = |\mathscr{T}| + (d+1-j)j$.*

3.2 Corollary. *For an arbitrary enumeration $w_1, w_2, \cdots, w_k$ of the interior points of P, there exists a refinement sequence*

$$\mathscr{T}_1 \prec \mathscr{T}_2 \prec \cdots \prec \mathscr{T}_k \tag{9}$$

such that $\mathscr{T}_1 = \mathscr{T}_{w_1}$, $\mathscr{V}(\mathscr{T}_i) = \mathscr{V}(\mathscr{T}_{i-1}) \cup \{w_i\}$ for $2 \leq i \leq k$, and $|\mathscr{T}_k| \geq dk+1$. Moreover, $\operatorname{Vol}(P) \geq \frac{1}{d!}(dk+1)$.

Bey et al. prove (8) by decomposing P into subpolytopes and inducting on k. In their proof, they also note that equality occurs only if the subpolytopes are simplices, and for $k = 1$, equality in (8) occurs if and only if P is unimodularly equivalent to

$$S_d(k) = \operatorname{conv}\left(e_1, \cdots, e_d, -k\sum_{i=1}^{d} e_i\right), \tag{10}$$

where the e_i are the unit points. In [6], I prove that this is true for all $k \geq 1$ and $d \neq 2$, and give counterexamples for $d = 2$. Selected results follow below.

There is a subtle yet significant difference between using decompositions into general subpolytopes and triangulations. It is specifically through triangulations that I prove the following results.

3.3 Corollary. *In (9), $|\mathscr{T}_i| - |\mathscr{T}_{i-1}| > d$ if and only if w_i lies in the relative interior of a simplex in $\mathscr{T}_{i-1}$, or w_i lies in the relative interior of an edge of a simplex in $\mathscr{T}_{i-1}$.*

3.4 Theorem. *Suppose T is clean d-simplex ($d \geq 3$) with $k \geq 2$ interior lattice points. Let w_1 and w_2 be any two interior lattice points of T, and let $\mathscr{T}_{w_1} = \{T_i\}$ be the basic triangulation of T with respect to w_1. If w_2 lies in the relative interior of a simplex in $\mathscr{T}_{w_1}$, say T_n, then there exist $\alpha_i \in \mathbb{R}$ such that*

$$w_2 = \alpha_1 w_1 + \alpha_n v_n + \sum_{\substack{i=2\\ i\neq n}}^{d+1} \alpha_i v_i, \qquad \sum_{i=1}^{d+1} \alpha_i = 1,$$

$\alpha_1 > 0$, $\alpha_n < 0$, and $\alpha_i \geq 0$ otherwise.

The geometric interpretation of Theorem 3.4 is that there exist two simplices in the refinement of $\mathscr{T}_{w_1}$ with respect to w_2 whose union forms a d-polytope having $d+2$ vertices containing a "j-bipyramid," where $2 \leq j \leq d$. A *d-bipyramid* is the convex hull of a line L and a $(d-1)$-simplex S, where $\operatorname{int}(L) \cap \operatorname{int}(S)$ is a single point. There is an extensive amount of literature on the properties of $d+2$ points in $\mathbb{R}^d$ not contained in a hyperplane (cf. [3], [8], [9], [10], [15], [17], [20], [22], [23], and [30]). C.L. Lawson's [17] paper, however, specifically deals with triangulations, and his results directly imply the following.

3.5 Theorem. *Let $\mathscr{P}$ be a set of $d+2$ points in $\mathbb{R}^d$ not contained in any hyperplane. If there exists a subset $\mathscr{P}' \subseteq \mathscr{P}$ such that $\operatorname{conv}(\mathscr{P}')$ is a j-bipyramid, where $2 \leq j \leq d$, then $\mathscr{P}$ has two triangulations, one with cardinality 2 and another with cardinality j.*

The results above essentially imply the following main result; 2-bipyramids (planar quadrialaterals) require special considerations.

3.6 Theorem. *If $d \neq 2$ and P is also clean, then $\operatorname{Vol}(P) = \frac{1}{d!}(dk+1)$ if and only if P is unimodularly equivalent to $S_d(k)$.*

Interestingly, Theorem 3.6 fails in dimension 2. Note that the interior points of $S_d(k)$ are collinear with vertex $-k(e_1+e_2+\cdots+e_d)$. In $\mathbb{R}^2$, there are clean triangles which do not have this property. Consider the family of triangles $\Delta(p,q,r)$ with vertex set

$$\{\ (-r,0),\ (0,q),\ (p,-1)\ \},$$

where p, q, and r are positive integers and $\gcd(r,q) = \gcd(p,q+1) = 1$. These triangles are clean, and satisfy the minimal volume condition by Pick's Theorem. However, the interior lattice points are covered by the vertical lines $x = i$, where

$-r < i < p$. Since the covering lines are parallel, $\Delta(p, q, r)$ cannot be unimodularly equivalent to a triangle whose interior points are collinear except when $r = p = 1$.

Finally, I would like to mention a lower bound (in the spirit of Pick's Theorem) of a (not necessarily clean) lattice polyhedron.

3.7 THEOREM. *If P is a lattice polyhedron with b boundary lattice points and $k \geq 1$ interior lattice points, then*

$$\mathrm{Vol}(P) \geq \frac{2b + 3k - 7}{6}.$$

An equivalent result appears in [5] with a slightly different set of assumptions.

4. Number-Theoretic Interpretation

The problem of classifying clean k-point lattice tetrahedra can be stated in non-geometric terms.

PROBLEM. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. Given a positive integer k, find integer values of a, b, and n which satisfy

(1) $0 < a \leq b < n$
(2) $\gcd(a, n) = \gcd(b, n) = \gcd(c, n) = 1$, where $c = 1 - a - b$
(3) $k = \#\{\ t : 1 \leq t \leq n - 1 \text{ and } A_t = 1\ \}$, where

$$A_t := \left\{ \frac{t(n-a)}{n} \right\} + \left\{ \frac{t(n-b)}{n} \right\} + \left\{ \frac{t(n-c)}{n} \right\} + \left\{ \frac{t}{n} \right\}.$$

The case $k = 1$ has been completely resolved by A. Kasprzyk [13] and Reznick [27]. For $n = 3k + 1$, the corresponding values of a and b are

$$(a, b) \in \{\ (3, 3k),\ (3k, 3),\ (2k + 1, 2k + 1),\ (3k, 3k)\ \},$$

and can be determined using relatively elementary methods. One method relies on congruences determined by Reznick in [27]. Another method uses determinants and the fact that $S_d(k)$ is unimodularly equivalent to

$$T_{a_1,a_2,\cdots,a_d} = \mathrm{conv}\left(\mathbf{0},\ e_1,\ e_2,\ \cdots\ ,\ e_{d-1},\ \sum_{i=1}^{d} a_i e_i\right),$$

where $a_d = dk + 1$, as shown in [6]. The conjectured upper bound in (1) suggests that the largest value of n among all possible solutions is $12k + 8$.

5. Directions for Future Research

It seems unlikely that properties of triangulations alone will lead to a proof of the upper bound in (1). However, the last 50 years have given way to many new results (cf. [19], [28], [29]) relating to the combinatorics of simple and simplicial polytopes). Many new tools (cf. [1], [7]) have also been developed for lattice point enumeration.

EHRHART THEORY. As already noted, k_{d-i} in equation (7) can be computed using Ehrhart polynomials. More precisely, if $i_P(n)$ is the Ehrhart polynomial of a lattice d-polytope P, then $k_n + b_n = i_P(n)$ and $k_n = (-1)^d \cdot i_P(-n)$. For a clean, k-point lattice tetrahedron T,

$$i_T(n) = \mathrm{Vol}(T) \cdot n^3 + n^2 + (k + 2 - \mathrm{Vol}(T)) \cdot n + 1.$$

Hence

$$k_n(T) = \mathrm{Vol}(T) \cdot n^3 - n^2 + (k + 2 - \mathrm{Vol}(T)) \cdot n - 1.$$

Question. Can the coefficients of k_n be perturbed in some way to obtain an improved upper bound for the volume of tetrahedra?

f-vectors and h-vectors. In 1998, Kantor [12] showed that if $\mathscr{T}$ is a full triangulation of P, then

$$i_P(n) = \mathrm{Vol}(P) \cdot n^3 + \left(\frac{f_2}{2} - f_3\right) \cdot n^2 + \left(f_1 - \frac{3}{2}f_2 + 2f_3 - \mathrm{Vol}(P)\right) \cdot n + 1,$$

where the f_i are the components of the "f-vector" of $\mathscr{T}$. The f-vector of a $(d-1)$-dimensional simplicial complex $\mathcal{C}$ is the vector $f(\mathcal{C}) = (f_0, f_1, \cdots, f_{d-1})$ where f_i counts the number of i-faces in $\mathcal{C}$. (Note that triangulations can be viewed as simplicial complexes.) The h-vector $h(\mathcal{C}) = (h_0, h_1, \cdots, h_d)$ is defined by the relation

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}$$

with the understanding that $f_{-1} = 1$. Since it is known that

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i,$$

an upper bound for the components of the h-vector naturally gives an upper bound for the components of the f-vector. P. McMullen [19] gave sharp upper bounds for h_j for simplicial d-polytopes with n vertices.

Question. Can the bound on h_j be extended to simplicial complexes?

I hope to gain some insight on the upper bound in (1) through the ideas above in conjunction with properties of triangulations.

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