NAME:

Math 395C – Honors Linear Algebra
Prof. Ward Henson
Final Exam – December 17, 2004

Problem:	1	2	3	4	5	6
Points	20	20	20	20	20	20
Score:						

Problem:	7	8	9	10	Total
Points	20	20	20	20	200
Score:					

There are ten (10) problems on this Exam and you should do all of them. They are weighted as indicated.

To receive full credit, each of your solutions must be fully justified; give enough explanation of each solution to make it clear you understand what is going on. Where a proof is called for, be as clear and complete as you can be when writing your argument.

In justifying your proofs, you may cite any result from the textbook or proved in class; however, you may not cite homework problems. Note that some problems have stricter requirements than this.

Problem 1. (20 points) Fix $n \geq 1$. Let $\mathcal{C}_n$ be the set of all $n \times n$ matrices $A = (a_{ij})$ over $\mathbb{R}$ whose entries all satisfy $a_{ij} \in \{0, 1, -1\}$. For each $A \in \mathcal{C}_n$ let $Z(A)$ be the number of entries of A that equal 0. For $A \in \mathcal{C}_n$ such that $\det(A) \neq 0$, what is the minimum possible value of $Z(A)$?

Problem 2. (20 points) Which $n \times n$ matrices over $\mathbb{R}$ satisfy the following condition: for every $n \times n$ permutation matrix P, one has $AP = PA$.

Problem 3. (20 points) Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. For $A, B \in V$ define $\langle A, B \rangle = \mathrm{tr}(AB)$, where tr denotes the trace (sum of the diagonal entries). It is easy to show that this is a symmetric, bilinear form on V (and you can use this fact without proof).
(a) (10 points) Show that this form is *nonsingular* on V. That is, if $A \in V$ and $A \neq 0$, show that there exists $B \in V$ such that $\langle A, B \rangle \neq 0$.
(b) (10 points) Show that V has a basis $A_1, \ldots, A_n$ which is *orthogonal* with respect to this form; that is, such that $\langle A_i, A_j \rangle = 0$ whenever $i \neq j$. (You can either exhibit such a basis explicitly or give an abstract proof that it has to exist.)

Problem 4. (20 points) Let F be a field and let G be the set of all 2×2 matrices over F that have the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

where $a \neq 0$.
(a) (10 points) Prove that matrix multiplication defines a law of composition on G under which it is a group.
(b) (10 points) Under what conditions is the group G abelian?

Problem 5. (20 points) Let $n \geq 1$ and let A be an $n \times n$ orthogonal matrix over $\mathbb{R}$.
(a) (7 points) Show that $\det(A)$ is either 1 or -1.
(b) (7 points) Let c be an eigenvalue of A; show that c is either 1 or -1.
(c) (6 points) When n is even, show that there is an $n \times n$ orthogonal matrix over $\mathbb{R}$ that has no eigenvalues.

Problem 6. (20 points) Let p be a prime and let V be a vector space of dimension 2 over the field $\mathbb{F}_p$ of p elements.
(a) (7 points) How many elements does V have?
(b) (13 points) How many subspaces does V have?

Problem 7. (20 points) Let V be a vector space over the field F with $\dim(V) = n$. Let $T \colon V \to V$ be a linear transformation such that $T^2 = T$ and T has rank m; assume $0 < m < n$. Show that there is a basis $\mathbb{B} = (v_1, \ldots, v_n)$ for V such that $T(v_j) = v_j$ for $j = 1, \ldots, m$ and $T(v_j) = 0$ for $j = m + 1, \ldots, n$.

Problem 8. (20 points) Let $c_1, \ldots, c_n$ be distinct elements of a field F and let $p(t)$ be the polynomial over F given by $p(t) = (t - c_1) \cdots (t - c_n)$.

(a) (7 points) Show that there is an $n \times n$ matrix A over F whose characteristic polynomial is $p(t)$.
(b) (13 points) Suppose A is any $n \times n$ matrix over F whose characteristic polynomial is $p(t)$. For each $j = 1, \ldots, n$ let B_j be the matrix $A - c_j I$, where I is the $n \times n$ identity matrix. Show that the matrix product $B_1 \cdots B_n$ is the 0 matrix.

Problem 9. (20 points) In this problem we work over the field $\mathbb{C}$ of complex numbers. Recall the Fundamental Theorem of Algebra: every polynomial of degree ≥ 1 over $\mathbb{C}$ has a root in $\mathbb{C}$.
Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T\colon V \to V$ be a linear operator. For each $j \geq 1$ let $W_j = \mathrm{im}(T^j)$.
(a) (5 points) For each $j \geq 1$, show that W_j is a subspace of V, that W_j is T-invariant, and that $W_j \supseteq W_{j+1}$.
(b) (15 points) Assume that 0 is the only eigenvalue of T. Show that for some $j \geq 1$, $W_j = \{0\}$.

Problem 10. (20 points) Let F be a field and let W be a vector space over F. Let X and Y be finite dimensional subspaces of W and set $U = X \cap Y$ and $V = \mathrm{span}(X \cup Y)$.
(a) (7 points) Show that the subspaces U and V are finite dimensional.
(b) (13 points) What is the relationship among the numbers $\dim(X), \dim(Y), \dim(U), \dim(V)$? (With proof.)