Math 395C — Honors Linear Algebra — Prof. Henson

Final Exam — 12/17/04 — Sketches of Solutions

$\boxed{1}$ There are $n\times n$ matrices A with $\det A \neq 0$ and $Z(A)=0$. For example, put 1 on the main diagonal and above it, put -1 below. The n^{th} column has all 1s. Add it to the other $n-1$ columns. The result is an upper triangular matrix with diagonal entries 2 ($n-1$ times) and 1 (1 time). So its det is $2^{n-1} \neq 0$.

$\boxed{2}$ Let J be the matrix of all 1s and I the identity. It's easy to see that $PJ = J = JP$ and $PI = P = IP$ for every permutation matrix P. Hence for any real numbers a and b, $A = aI + bJ$ satisfies $PA = aP + bJ = PA$ for all P. A has all diagonal entries $= a+b$ and all off diagonal entries $= b$.

Careful use of specific permutation matrices can show that every matrix A that has $AP = PA$ for all P is of this form. In general, let P be the permutation matrix corresponding to the permutation $p \in S_n$. For $A = (a_{ij})$ the equation $PA = AP$, which is equivalent to $P^{-1}AP = A$, yields that $e_{p(i)p(j)} = e_{ij}$ for all i,j. By varying p we see $e_{KK} = e_{ii}$ holds for all i,k and that $e_{Kl} = e_{ij}$ for all i,j,k,l having $i\neq j$ and $K \neq l$.

One way to analyze the equation $P^{-1}AP = A$ is to note that $P^{-1}e_{ij}P = e_{p(i)p(j)}$ for all i,j.

(3 people got this problem completely right — congratulations!)

$\boxed{3}$ (a) If $A=(a_{ij})$ and $B=(b_{ij})$ it's easy to compute $tr(AB) = \sum_{i,j} a_{ij} b_{ji}$. Hence $tr(AA^t) = \sum_{i,j} (a_{ij})^2$ which is >0 if $A \neq 0$. Or if $a_{ij} \neq 0$, we have $tr(Ae_{ji}) = a_{ij} \neq 0$

(b) One such basis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$\boxed{4}$ Say $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ and $h = \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}$ are in G. Then $gh = \begin{bmatrix} ac & ad+b \\ 0 & 1 \end{bmatrix}$ which is in G since $ac \neq 0$ in F.

Matrix mult. is known to be associative and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$. is known to be an identity for it. Finally, h is an inverse of g iff $ac = 1$ and $ad+b = 0$, which can be solved for c and d because $a \neq 0$.

For G to be abelian we need $gh = hg$ for all $g, h \in G$. That is, $ad+b = cb+d$ for all $a,b,c,d \in F$ with $a, c \neq 0$. This equation is equivalent to $(a-1)d = (c-1)b$ and its obviously true if $a=c=1$. So G is abelian if $F = \{0,1\}$. Otherwise, take $a=1$ and $c \in F \setminus \{0,1\}$; this yields $0 = (c-1)b$ for all $b \in F$. Taking $b=1$ gives a contradiction.

$\boxed{5}$ (a) $1 = \det(AA^{-1}) = \det(AA^t) = \det(A)^2$

(b) Say $X \neq 0$ and $AX = cX$. Then $c^2(X \cdot X) = (cX \cdot cX) = (AX \cdot AX) = (X \cdot X)$; hence $c^2 = 1$ since $(X \cdot X) \neq 0$

(c) Put $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ k times down the diagonal, $n = 2k$.

6 (a) V is isomorphic to $\mathbb{F}_p^2$, which has p^2 elements.

(b) Every vector in $\mathbb{F}_p^2$ is a scalar multiple of one of the following vectors: $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \ldots, \begin{bmatrix}1\\p-1\end{bmatrix}$

and none of these is a scalar multiple of any other. So $\mathbb{F}_p^2$ has $p+1$ subspaces of $\dim = 1$ and $p+3$ subspaces in all.

7 The dimension formula yields $\ker T$ to have $\dim = n-m$. Also $\ker T \cap \operatorname{im} T = \{0\}$. Hence V is the direct sum of $\ker T$ and $\operatorname{im} T$. For $v \in \operatorname{im} T$, $Tv = v$.

8 (a) Take the diagonal matrix D with diagonal entries $c_1, \ldots, c_n$

(b) Since there is a basis of eigenvectors for such an A, we have $A = PDP^{-1}$ for some invertible P. Then $(A - c_1 I) \cdots (A - c_n I) = P(D - c_1 I) \cdots (D - c_n I) P^{-1}$ so it suffices to show $(D - c_1 I) \cdots (D - c_n I) = 0$. Note that row i is 0 in $(D - c_i I)$ and hence ~~for~~ row i is 0 in $(D - c_1 I) \cdots (D - c_n I)$. For all i !!

9 (a) T^j is linear so $\operatorname{im}(T^j)$ is a subspace. For any $v \in V$, $T(T^j v) = T^{j+1}(v) = T^j(Tv)$ so $W_{j+1} \subseteq W_j$ and T maps W_j into W_{j+1}.

(b) Since V is finite dimensional, $W_m = W_{m+1}$ for large enough m. So T maps W_m onto itself. This contradicts the assumptions if $W_m \neq \{0\}$. (Over $\mathbb{C}$ we'd get an eigenvector for T in W_m; it would have to be in $\ker T$. But this would force $\dim(W_{m+1}) < \dim(W_m)$ by the dimension formula.)

$\boxed{10}$ $\dim X + \dim Y = \dim \underbrace{X \cap Y}_{U} + \dim \underbrace{(X + Y)}_{V}$

The proof is on page 103