

Math 411: Model Theory
Spring Semester 2004
Prof. Ward Henson

Problem Set 5 – Due 4:30pm, Friday, May 7, 2004.

(Revised later to add countability hypotheses in 5.3)

There are five problems (equally weighted) and you should do four of them. To earn full credit requires a careful writeup of each problem, taking care to justify everything you claim and to explain your ideas clearly.

5.1. PROBLEM. Consider the language L_r of rings with 1. Let R be any ring with 1 and let $M_2(R)$ be the ring of 2×2 matrices over R with the identity matrix as its 1. Considering both rings as L_r -structures, let $T = \text{Th}(R)$ and $T_2 = \text{Th}(M_2(R))$. In each part below, is the given implication true for all rings R ?

- (a) If T is ω -categorical, then T_2 is ω -categorical.
- (b) If T is uncountably categorical, then T_2 is uncountably categorical.
- (c) If T is ω -stable, then T_2 is ω -stable.

5.2. PROBLEM. Let L be the language with countably many unary predicates $(P_n \mid n \in \mathbb{N})$ and no other nonlogical symbols. Let T be the L -theory axiomatized by the sentences asserting that $P_1 \supseteq P_2 \supseteq P_3 \supseteq \dots$, and that the complement of P_1 and each $P_n \setminus P_{n+1}$ are infinite. Show that T is complete and calculate the Morley rank and degree of the formula $x = x$ in models of T .

5.3. PROBLEM. Let L be countable and let \mathcal{A} be any countable L -structure. Show that there exists a sequence $(\mathcal{B}_n \mid n \in \mathbb{N})$ of L -structures such that (i) each \mathcal{B}_n is a proper elementary extension of \mathcal{A} , (ii) $\mathcal{B}_{n+1} \preceq \mathcal{B}_n$ for each $n \in \mathbb{N}$, and $A = \bigcap \{B_n \mid n \in \mathbb{N}\}$. (Hint: take \mathcal{B}_1 to be a Skolem hull generated by A and a suitable sequence of indiscernibles.)

5.4. PROBLEM. Suppose T is a complete, ω -stable theory in a countable language L . Suppose $\mathcal{A} \models T$ and let $\varphi(x)$ be an $L(\mathcal{A})$ -formula whose Morley rank is the ordinal α . Show that there exists an elementary extension \mathcal{B} of \mathcal{A} such that as $\psi(x)$ varies over all $L(\mathcal{B})$ -formulas, the Morley rank of $\varphi \wedge \psi$ varies over all ordinals $\beta \leq \alpha$.

5.5. PROBLEM. For each $n \in \mathbb{N}$, give an example of a complete ω -stable theory T_n in a countable language L_n such that the Morley rank of the formula $x = x$ in models of T_n is exactly n .