Math 571: Model Theory
Spring Semester 2008
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Some exercises for sections 10 and 11
Also see all the exercises that are implicit in the lecture notes, especially the "Facts".

10.1. Let L be the language of pure equality and let T be the theory in L of all infinite sets. From Example 3.16 we know that T admits QE and is complete.
• Show that T is strongly minimal.
• Explain the meaning of the dimension of a given model of T, in the sense of Section 10.

10.2. Let L be the language whose nonlogical symbols consist of a unary function symbol F. Let T be the theory in L of the class of all L-structures (A, f) in which f is a bijection from A onto itself and f has no finite cycles. From Problem 2.2 we know that T admits QE and is complete. Note that $(\mathbb{Z}, S)$ is a model of T, where $S(a) = a + 1$ for all $a \in \mathbb{Z}$; therefore $T = \mathrm{Th}(\mathbb{Z}, S)$.
• Show that T is strongly minimal.
• Explain the meaning of the dimension of a given model of T, in the sense of Section 10.

10.3. Let K be a field and let L be the language of vector spaces over K. Let T be the theory in L of all infinite vector spaces over K. (See Exercises 3.6, 5.4, and 9.3.)
• Show that T is strongly minimal.
It follows that Section 10 applies to infinite K-vector spaces. Exercise 9.3 shows that algebraic closure in the sense of model theory and linear span in the sense of linear algebra are identical, when applied to subsets of a fixed infinite vector space over K.
• Let V, W be infinite K-vector spaces and let $X \subseteq V, Y \subseteq W$ be K-linear subspaces. Suppose $F\colon X \to Y$ is a K-linear isomorphism. Show that F is an elementary map in the sense of the L-structures V, W. (Note that if K is a finite field, and X, Y are finitely generated, then they are not models of T.)
• If V is an infinite K-vector space and $X \subseteq V$ is a K-linear subspace, show that the model theoretic dimension of X in the sense of algebraic closure in V does not depend on V. Show that this dimension is the same as the dimension of X in the sense of linear algebra.
• Check that Theorem 10.3 implies all of the standard facts about linearly independent sets, spanning sets, and bases, for arbitrary vector spaces over K.

10.4. Let T be a strongly minimal L-theory and let κ be an infinite cardinal. Let $\mathcal{A}$ be an infinite model of T.

• Show that $\mathcal{A}$ is κ-saturated iff the dimension of $\mathcal{A}$ in the sense of Section 10 is $\geq \kappa$.

10.5. Let L be the language whose only nonlogical symbol is a binary predicate symbol $<$. Let $\mathcal{A}$ be any infinite linear ordering, considered as an L-structure.

• Show that $\mathrm{Th}(\mathcal{A})$ is not strongly minimal.

11.1. Let K be a countable ordered field, considered as an L_{or}-structure, and let $T = \mathrm{Th}(K)$. Show that there exists a 1-type $p \in S_1(T)$ that is not realized in K. Therefore, no countable ordered field is ω-saturated.

11.2. Let R be an ordered field. Let x be a transcendental element over R and consider the field $R(x)$ of rational functions in x with coefficients in R.

• Show that there is linear ordering $<$ on $R(x)$ that makes $R(x)$ into an ordered field, such that $r < x$ for all $r \in R$.

• Show that this ordering is unique.

• Show how to embed the ordered field $R(x)$ with this ordering into a suitable ultrapower of R.

• Describe all the embeddings of the field $R(x)$ into an ultrapower of R. (Each one induces a field ordering on $R(x)$.)

11.3. Use the preceding Exercise and results in Section 11 to show that the theory $RCOF$ is not κ-categorical for any infinite cardinal κ. (For example, construct models of $RCOF$ of cardinality κ, such that one has an ordering of cofinality ω and the other has an ordering of uncountable cofinality.)