Math 571: Model Theory
Spring Semester 2008
Prof. Ward Henson

Problem Set 5
Due in class Monday, April 14

There are four problems (equally weighted) and you should do all of them. To earn full credit requires a careful writeup of each problem, taking care to justify everything you claim and to explain your ideas clearly. Do not look up solutions in textbooks or other sources, and make sure your submitted solutions are your own work.

5.1 Problem. Let L be the language whose only nonlogical symbol is the binary predicate symbol E. Let $\mathbb{K}$ be the class of L-structures in which E is interpreted by an equivalence relation that has precisely one equivalence class of cardinality n for each positive integer n, and any number of infinite equivalence classes. For each $\mathcal{A}$ in $\mathbb{K}$, let A_f be the union of the finite equivalence classes of $E^{\mathcal{A}}$. Let T be the theory of $\mathbb{K}$. By Problem 1.1, $\mathbb{K} = \mathrm{Mod}(T)$, a fact you may use in solving this problem.

Let L' be the extension of L obtained by adding unary predicates P_n for each $n \geq 1$. Let T' be the L'-theory obtained from T by adding for each $n \geq 1$ the sentence expressing the condition "$P_n(x)$ if and only if the equivalence class of x has cardinality n". Note that each model of T has an expansion that is a model of T', so T' is a conservative extension of T.

- Show that T' admits quantifier elimination.
- Use the preceding item to show that T is complete.
- Show that if $\mathcal{A} \subseteq \mathcal{B}$ are both models of T, then $\mathcal{A} \preceq \mathcal{B}$ if and only if $A_f = B_f$.

5.2 Problem. Let L be the language whose nonlogical symbols are the unary predicates $(U_n \mid n \in \mathbb{N})$. For any finite $F, G \subseteq \mathbb{N}$, let $\sigma_{F,G}$ be the L-sentence

$$\exists x \left(\bigwedge_{j \in F} U_j(x) \wedge \bigwedge_{j \in G} \neg U_j(x) \right).$$

Let T be the L-theory axiomatized by the set of all sentences $\sigma_{F,G}$ where F, G are disjoint finite subsets of $\mathbb{N}$. This theory is known as the *theory of infinitely many independent sets.*

For each $n \in \mathbb{N}$, let L_n be the sublanguage of L whose nonlogical symbols are the predicates $U_0, \ldots, U_n$. Let T_n be the set of all L_n-sentences that are consequences of T. Note that every model of T is infinite, and hence the same is true of T_n.

- Show that each T_n admits QE and is complete.
- Use the previous item to show that T admits QE and is complete.
- Show that there is no principal type in $S_m(T)$, for each $m \geq 1$.

5.3 Problem. Let F be an ordered field. Suppose that for any formula $\varphi(x)$ in the language of ordered rings for which $F \models \exists x \varphi(x)$, there is a finite element a of F such that $F \models \varphi[a]$. (Here x is a single variable.)

• Show that there is an Archimedean ordered field K that is elementarily equivalent to F.

(An element of an ordered field is *finite* if it is bounded above in absolute value by some element of the prime field. An ordered field is *Archimedean* if all of its elements are finite. Note that an ordered field is Archimedean if and only if it is isomorphic to an ordered subfield of $\mathbb{R}$.)

5.4 Problem. Let T be a complete theory in a countable language, with no finite models. Show that there exists a countable model $\mathcal{A}$ of T such that $\mathcal{A}$ is isomorphic to a proper elementary substructure of itself. (Hint: take $\mathcal{A}$ to be a Skolem hull generated by a suitable sequence of indiscernibles.)