ADDENDUM

We add a few more set theoretic items that will later be incorporated into Section 2.

Here we assume X is a set of individuals that contains $\mathbb{N}$ as a subset. In the sequel, we only consider superstructures that satisfy this condition. Since we are already assuming that X is infinite, this additional condition amounts to specifying a distinguished countable subset C of X and a linear ordering $<$ on C that makes $(C, <)$ isomorphic to $(\mathbb{N}, <)$. Note that C and the binary relation $<$ are elements of $U(X)$.

We formalize n-tuples in superstructures in a way that keeps the rank bounded as n varies. For each $a_1, \dots, a_n$ in $U(X)$, we define the n-tuple of $a_1, \dots, a_n$ by

$$\begin{aligned}(a_1, \dots, a_n) &= \{\langle 1, a_1\rangle, \dots, \langle n, a_n\rangle\}\\ &= \{\{\{1\}, \{1, a_1\}\}, \dots, \{\{n\}, \{n, a_n\}\}\}.\end{aligned}$$

Equivalently, if f is any function in $U(X)$ whose domain is $\{1, \dots, n\}$, we regard f as identical to the n-tuple $(f(1), \dots, f(n))$.

The key property of this construction is that the rank of $(a_1, \dots, a_n)$ only depends on the ranks of its coordinates and not on n:

$$\operatorname{rank}((a_1, \dots, a_n)) = \max(\operatorname{rank}(a_1), \dots, \operatorname{rank}(a_n)) + 3.$$

Therefore, if A is a set in $U(X)$, then there exists a set B in $U(X)$ that consists of all n-tuples of elements of A (with n varying over $\mathbb{N}$); if A is nonempty, then the rank of B equals $\operatorname{rank}(A) + 3$.

In using n-tuples later we will need Δ_0-formulas like the following:

(19) "f is an n-tuple for some $n \in \mathbb{N}$" is expressed by the conjunction of the formula in item (14) and

$$\exists t \in \mathbb{N} \Big[t \neq 0 \ \wedge \ \forall v \in f \ \forall w \in v \ \forall z \in w \ \forall u \in w \ \big(v = \langle u, w\rangle \ \rightarrow \ u \in \mathbb{N}\big) \ \wedge \\ \forall u \in \mathbb{N} \ \big(\exists v \in f \ \exists w \in v \ \exists z \in w \ (v = \langle u, z\rangle) \ \leftrightarrow \ 0 < u < t\big)\Big].$$

Note that here we need to express "$u < t$" by a Δ_0-formula such as

$$\exists z \in \, < (z = \langle u, t\rangle).$$

(20) "f is a finite tuple whose coordinates come from the set x" is expressed by the conjunction of $\neg I(x)$ and the formula in item (19) and

$$\forall y \in f \ \forall w \in y \ \forall u \in w \ \forall v \in w \ (y = \langle u, v\rangle \ \rightarrow v \in x).$$

In the previous two items our formulas used constants for the elements 0, $\mathbb{N}$ and $<$ of our superstructure. In general it is sometimes necessary (as here) and often useful to allow such constants in the Δ_0-formulas we use.

(21) Fix $n \in \mathbb{N}$. The condition "x is a set of rank $\leq n+1$ and y is the transitive closure of x" is expressed by the conjunction of $\neg I(x) \wedge \neg I(y)$ and "x has rank $\leq n+1$" and "y has rank $\leq n+2$" and "y is transitive" and $x \subseteq y$ (using the formulas in items (1), (17) and (18)) and

$$\forall w \in U_{n+2}(X)\ \big((\neg I(w) \wedge w \text{ is transitive } \wedge x \subseteq w)\ \rightarrow y \subseteq w\big).$$

Next we present another aspect of substructures of $U(X)$ that will be used in the next section. Our main purpose is to give a simple isomorphic characterization of transitive substructures of superstructures.

2.7. **Definition.** An L-structure $\mathcal{A}$ is *strongly well-founded* if for each element a of $\mathcal{A}$ there exists $n \geq 0$ such that whenever we have $b_1, \ldots, b_k$ in $\mathcal{A}$ with $b_1 E^{\mathcal{A}} a$ and $b_{j+1} E^{\mathcal{A}} b_j$ for all $j = 1, \ldots, k-1$, then it must be true that $k \leq n$.
Given a in a strongly well-founded L-structure $\mathcal{A}$, we will call the smallest n for which this condition holds the *height* of a in $\mathcal{A}$.
We call an L-structure $\mathcal{A}$ *well-based* if the set of elements of $\mathcal{A}$ having height 0 contains the set $I^{\mathcal{A}}$.

Note that any substructure $\mathcal{A}$ of a superstructure $U(X)$ is strongly well-founded and well-based. Indeed, the height of any a in such an $\mathcal{A}$ is bounded above by the rank of a in $U(X)$. If $\mathcal{A}$ is a strongly well-founded L-structure, a, b are elements of $\mathcal{A}$, and $bE^{\mathcal{A}}a$, then the height of b in $\mathcal{A}$ is strictly less than the height of a in $\mathcal{A}$.

Note also that the property of being well-based can be expressed by the following L-sentence, which is the universal closure of a Δ_0-formula.

$$\forall x\ \big(I(x)\ \rightarrow\ \neg \exists y \in x\ (y = y)\big)\ .$$

The *Extensionality Axiom* in set theory asserts that two sets with the same elements are equal. This can be expressed by the following L-sentence:

$$\forall x\ \forall y\ \big[\big(\neg I(x) \wedge \neg I(y) \wedge \forall u \in x\ (u \in y) \wedge \forall u \in y\ (u \in x)\big)\ \rightarrow\ x = y\big]$$

Note that this sentence is also the universal closure of a Δ_0-formula. Moreover, this sentence is obviously true in every transitive substructure of a superstructure.

The following result shows that a version of the converse of the previous observations is also true. The method of proof goes by the name "Mostowski collapse".

2.8. **Proposition.** *Let $\mathcal{A}$ be an L-structure. If $\mathcal{A}$ is strongly well-founded, is well-based, and satisfies the Extensionality Axiom, then there is a transitive substructure $\mathcal{B}$ of $U(I^{\mathcal{A}})$ and an isomorphism G of $\mathcal{A}$ onto $\mathcal{B}$ such that G is the identity function on $I^{\mathcal{A}}$. Moreover, G (and therefore also $\mathcal{B}$) is uniquely determined by $\mathcal{A}$.*

Proof. We define $G\colon \mathcal{A} \to U(I^{\mathcal{A}})$ on elements of $\mathcal{A}$ by induction on their height in $\mathcal{A}$. On the elements of $I^{\mathcal{A}}$ we take G to be the identity. If $a \in \mathcal{A}$ does not satisfy $I^{\mathcal{A}}$ and also does not satisfy $\exists x \in a\ (x = x)$, then a is unique, since $\mathcal{A}$ satisfies the Extensionality Axiom, and we set $G(a) = \emptyset$. This defines G on all elements of $\mathcal{A}$ having height 0.

If G has been defined on all elements of $\mathcal{A}$ that have height $\leq n$ in $\mathcal{A}$ and $a \in \mathcal{A}$ has height $n+1$ in $\mathcal{A}$, we define

$$G(a) = \{G(b) \mid b \in \mathcal{A} \wedge bE^{\mathcal{A}}a\}.$$

As we proceed through this induction, we must check at each stage that there is a uniform bound on the ranks of $G(b)$ as b ranges over the elements of a in $\mathcal{A}$, so the set $\{G(b) \mid b \in \mathcal{A} \wedge bE^{\mathcal{A}}a\}$ is in $U(I^{\mathcal{A}})$. Indeed, it is easy to see that G defined in this way has the property that for any $bE^{\mathcal{A}}a$ we have

$$\operatorname{rank}(G(b)) \leq \operatorname{height}(b) < \operatorname{height}(a).$$

It is now routine to check by induction on height that G is an isomorphism onto a transitive substructure of $U(I^{\mathcal{A}})$. The assumption that $\mathcal{A}$ satisfies the Extensionality Axiom is used to prove that G is injective.

Likewise, when we have two isomorphisms G_1, G_2 from $\mathcal{A}$ onto transitive substructures of $U(I^{\mathcal{A}})$, and we know that $G_1(x) = G_2(x) = x$ for every individual x of $\mathcal{A}$, then we may prove $G_1(a) = G_2(a)$ for all $a \in \mathcal{A}$ by induction on the height of a in $\mathcal{A}$. $\square$