3. Nonstandard Extensions

This section is a continuation of the previous one. Here we introduce *nonstandard extensions* of superstructures, which are the basis for nonstandard analysis as we will present it.

3.1. **Definition.** Let X be an infinite set of individuals. A *nonstandard extension* of the superstructure $U(X)$ over X is a mapping between superstructures

$$^*\colon U(X) \to U(Y)$$

with the following properties:
(i) Y is a set of individuals containing X as a subset, Y is the value of the * mapping when applied to the element X of $U(X)$, and * is the identity map when applied to elements of X.
(ii) The mapping * is elementary for Δ_0-formulas; that is, for any Δ_0-formula $\varphi(x_1, \ldots, x_m)$ and any $a_1, \ldots, a_m \in U(X)$,

$$\mathcal{U}(X) \models \varphi[a_1, \ldots, a_m] \text{ if and only if } \mathcal{U}(Y) \models \varphi[{}^*a_1, \ldots, {}^*a_m].$$

Further, such a nonstandard extension is *proper* if there exists a countable, infinite set $C \subseteq X$ such that $^*C \setminus C \neq \emptyset$.

3.2. **Notation.** If $^*\colon U(X) \to U(Y)$ is a nonstandard extension and a is an element of $U(X)$, we write $^*a$ for the value of this mapping at argument a.

3.3. **Terminology.** When we are using a nonstandard extension and we invoke clause (ii) of Definition 3.1, we say that we are using the *Transfer Principle* or that we are arguing *by transfer*.

The existence of proper nonstandard extensions is proved using tools from elementary model theory, such as the compactness theorem or the ultrapower construction. We assume the reader knows how to construct elementary extensions of a given structure.

3.4. **Theorem** (Existence of nonstandard extensions)**.** *Let X be any infinite set of individuals. There exists a proper nonstandard extension of $U(X)$.*

Proof. Recall that we consider $U(X)$ as a structure $\mathcal{U}(X)$ equipped with the membership relation and with X as a distinguished subset. The associated language L has a binary predicate E and a unary predicate I.

Using the compactness theorem or the ultrapower construction from elementary model theory, we obtain an L-structure $\mathcal{M} = (M, E^{\mathcal{M}}, I^{\mathcal{M}})$ and an elementary embedding F of $\mathcal{U}(X)$ into $\mathcal{M}$. Without loss of generality, we assume that M_0 is a set of individuals that contains X as a subset, and that F is the identity map on X. (Change $\mathcal{M}$ to an isomorphic structure if necessary.)

Fix a countable, infinite subset C of X. To ensure that our eventual nonstandard extension is proper, we take $\mathcal{M}$ and F to also have the following property: there exists an element u of M such that $uE^{\mathcal{M}}F(C)$ while $u \neq F(c)$ holds for all $c \in C$. Since C has infinitely many elements, this is easy to achieve when we apply the compactness theorem or the ultrapower construction to obtain $\mathcal{M}$ and F.

We now define a family $(M_n \mid n \in \mathbb{N})$ of subsets of M by induction on n. To begin, let $M_0 = I^{\mathcal{M}}$. For each $n \in \mathbb{N}$, we define M_{n+1} from M_n by

$$M_{n+1} = \{u \in M \mid \forall v \in M(vE^{\mathcal{M}}u \ \rightarrow \ v \in M_n\}.$$

Finally, we let M_f be the union of the sets M_n as n ranges over $\mathbb{N}$, and we take $\mathcal{M}_f$ to be the substructure of $\mathcal{M}$ whose underlying set is M_f. This definition makes it clear that $\mathcal{M}_f$ is a transitive substructure of $\mathcal{M}$ and that $\mathcal{M}_f$ is strongly well-founded and well-based. (Note that if $u \in M_n$ and $vE^{\mathcal{M}}u$, then $n \geq 1$ and $u \in M_{n-1}$.) Furthermore, $\mathcal{M}$ satisfies the Extensionality Axiom, since $\mathcal{M}$ is elementarily equivalent to the superstructure $U(X)$, and thus $\mathcal{M}_f$ also satisfies the Extensionality Axiom.

Using the fact that F is an elementary embedding, it is easy to prove the following properties of $(M_n \mid n \in \mathbb{N})$; we leave the proofs to the reader.
(i) $M_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$.
(ii) The range of F is contained in M_f; indeed, for each $n \in \mathbb{N}$, if $x \in U_n(X)$ then $F(x) \in M_n$.

From (ii) and Proposition 2.5 it follows that, as a map from $\mathcal{U}(X)$ into $\mathcal{M}_f$, the embedding F preserves the meaning of Δ_0-formulas applied to elements of $U(X)$.

Next we use Proposition 2.8 to obtain an embedding G from $\mathcal{M}_f$ into the superstructure $U(M_0)$ with G the identity mapping on M_0.

We denote by $\mathcal{I}$ the transitive substructure of $\mathcal{U}(M_0)$ whose underlying set is the range of G. Using Proposition 2.5 it follows that, as a map from

$\mathcal{M}_f$ into $\mathcal{U}(M_0)$, the embedding G preserves the meaning of Δ_0-formulas applied to elements of M_f.

To complete the construction we take ${}^*\colon U(X) \to U(M_0)$ to be the composition of $G \circ F$. Putting together what was done above, we have that this defines a nonstandard extension of $U(X)$. (In particular, note that ${}^*X = G(F(X)) = M_0$ and that ${}^*x = x$ for every $x \in X$.)

To complete the proof we show that this nonstandard extension is proper. Indeed, let u be the element we included in M to satisfy the condition that $uE^{\mathcal{M}}F(C)$ while $u \neq F(c)$ holds for all $c \in C$. Recall that $C \subseteq X$, so ${}^*c = c$ for all $c \in C$. It follows that $G(u)$ is an element of ${}^*C = G(F(C))$ that is different from anything in C. □

Let ${}^*\colon U(X) \to U(Y)$ be a nonstandard extension. For any set A in $U(X)$, the image ${}^*A$ is a set in $U(Y)$; we think of ${}^*A$ as a *nonstandard extension* of A. This idea involves identifying A with $\{{}^*a \mid a \in A\}$. Indeed, whenever $a \in A$ we have ${}^*a \in {}^*A$ by transfer. We view the other elements of ${}^*A$ (if there are any) as "ideal elements" of A which have been added to facilitate mathematical arguments about structures based on A.

If A is finite, then it is easy to see that ${}^*A = \{{}^*a \mid a \in A\}$ and no elements have been added; this follows by applying the Transfer Principle to the Δ_0-formula

$$\forall x \in A \ (x = a_1 \vee \cdots \vee x = a_n).$$

However, if A is infinite and our nonstandard extension is proper, then ${}^*A$ does have elements other than ${}^*a$ for $a \in A$, as we show next.

3.5. **Proposition.** *Let X be an infinite set of individuals and suppose ${}^*\colon U(X) \to U({}^*X)$ is a proper nonstandard extension. If A is any infinite set in $U(X)$, then $\{{}^*a \mid a \in A\}$ is a proper subset of ${}^*A$.*

Proof. Since our nonstandard extension is proper, there exists a countable, infinite subset C of X for which C is a proper subset of ${}^*C$. Let $f\colon C \to A$ be a 1-1 function. By the Transfer Principle, ${}^*f$ is a 1-1 function from ${}^*C$ into ${}^*A$. (See the discussion of Δ_0-formulas in Section 2.) Let $u \in {}^*C \setminus C$. We show that ${}^*f(u) \in {}^*A \setminus \{{}^*a \mid a \in A\}$. Otherwise, there is some $a \in A$ satisfying ${}^*a = {}^*f(u)$. That is, in $U({}^*X)$ we have

$$\exists u \in {}^*C \ ({}^*a = {}^*f(u)).$$

By the Transfer Principle, it follows that in $U(X)$ we have

$$\exists c \in C\ (a = f(c)).$$

Then

$$^*f(u) = {}^*a = {}^*f(c),$$

while $u \neq c$, contradicting the fact that $^*f$ is 1-1. $\square$

Internal sets.

3.6. **Remark.** Consider a nonstandard extension $^*\colon U(X) \to U(^*X)$. Let $\varphi(x)$ be a Δ_0-formula in L and let a be a finite tuple of elements of $U(X)$. The Transfer Principle says that $\varphi(a)$ is true in $U(X)$ if and only if $\varphi(^*a)$ is true in $U(^*X)$. In order to use this principle effectively, we need to understand which objects in $U(^*X)$ are "referred to" when we interpret the statement $\varphi(^*a)$ in $U(^*X)$. Since φ is a Δ_0-formula, Proposition 2.5 shows us that these will all be elements of the transitive closure of $^*a$ in $U(^*X)$.

3.7. **Definition.** Let X be an infinite set of individuals and $^*\colon U(X) \to U(^*X)$ a nonstandard extension. Let $u \in U(^*X)$.
(i) u is *standard* if $u = {}^*a$ for some $a \in U(X)$.
(ii) u is *internal* if u is in the smallest transitive subset of $U(^*X)$ that contains all standard elements.
(iii) u is *external* if it is not internal.

3.8. **Proposition.** *Let X be an infinite set of individuals and $^*\colon U(X) \to U(^*X)$ a nonstandard extension. Let $u \in U(^*X)$. The following are equivalent:*
(i) u is internal.
(ii) $u \in {}^*a$ for some $a \in U(X)$.
(iii) $u \in {}^*U_n(X)$ for some $n \in \mathbb{N}$.

Proof. The implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are trivial.

If a is any set in $U(X)$, then $a \subseteq U_n(X)$, and hence $^*a \subseteq {}^*U_n(X)$, for some n. Hence the set of internal elements of $U(^*X)$ is the transitive closure of $\{^*U_n(X) \mid n \in \mathbb{N}\}$. Each $U_n(X)$ is transitive in $U(X)$. Transitivity of x is expressed by the Δ_0-formula $\forall y \in x\ \forall z \in y\ (z \in x)$. Hence each $^*U_n(X)$ is transitive in $U(^*X)$ by the Transfer Principle. This proves that (i) implies (iii). $\square$

3.9. **Exercises.** (Concerning the proof of 3.4.)
(i) Show that M_f is exactly the set of $E^{\mathcal{M}}$-well-founded elements of M. That is, for each $u \in M$ show that $u \notin M_f$ iff there is an infinite sequence $(u_n \mid n \in \mathbb{N})$ such that $u_0 = u$ and $u_{n+1} E^{\mathcal{M}} u_n$ for all $n \in \mathbb{N}$.
(ii) Show that $\mathcal{M}_f$ is *not* an elementary substructure of $\mathcal{M}$; in particular, M_f is a proper subset of M. (Hint: formalize the fact that for each natural number n there is in $U(X)$ a function f defined on $\{0, 1, \ldots, n\}$ and having the property that $f(k) \in f(k+1)$ holds for all $k < n$.)

3.10. **Exercise.** Let X be an infinite set of individuals and suppose $^*\colon U(X) \to U(^*X)$ is a proper nonstandard extension. Let A be an infinite set in $U(X)$. Show that the set $\{^*a \mid a \in A\}$ and its complement in $^*A$ are not internal. (Hint: first consider the case $A = \mathbb{N}$; use the Transfer Principle to show that if B is a nonempty internal subset of $^*\mathbb{N}$ then B has a least element under the linear ordering $^*\leq$. Use an argument similar to the proof of Proposition 3.5 to move from $\mathbb{N}$ to an arbitrary infinite set in $U(X)$.)