The rest of Section 3 is devoted to properties of standard and internal sets in a nonstandard extension.

For the rest of the section we take $X$ to be any infinite set of individuals and assume that $* : U(X) \to U(^*X)$ is a proper nonstandard extension.

We begin with some properties of the standard sets $^*U_n(X)$.

3.11. Proposition.
(1) $^*U_0(X) = ^*X = U_0(^*X)$.
(2) $^*U_n(X) \supseteq ^*U_m(X)$ for all $n \geq m \geq 0$.
(3) $^*U_n(X)$ is the set of all internal elements of $U(^*X)$ that have rank $\leq n$ in the superstructure $U(^*X)$, for each $n \geq 0$.
(4) $^*U_n(X)$ is a transitive set in $U(^*X)$, for each $n \geq 0$.
(5) Each element of $^*U_{n+1}(X) \setminus ^*U_n(X)$ is an internal set of rank $n + 1$ in $U(^*X)$, for each $n \geq 0$.

Proof. (1) This is immediate from the definitions.
(2) This is immediate from the Transfer Principle and the fact that $U_n(X) \supseteq U_m(X)$ when $n \geq m \geq 0$.
(3) Fix $n \geq 0$. By definition, every element of $^*U_n(X)$ is internal. Consider the $\Delta_0$-formula

$\forall u \in x \ (u \has\ rank \leq n)$.

This formula is true in $U(X)$ when $U_n(X)$ is substituted for $x$. By transfer it is true in $U(^*X)$ when $^*U_n(X)$ is substituted for $x$. This shows that every element of $^*U_n(X)$ has rank $\leq n$ in $U(^*X)$.

For the converse, let $v$ be an internal element of $U(^*X)$ that has rank $\leq n$. By Proposition 3.8, we have $v \in ^*U_k(X)$ for some $k$; by part (2) we may assume $k \geq n$. Consider the $\Delta_0$-formula

$\forall z \in y \ (z \has\ rank \leq n \to z \in x)$.

This formula is true in $U(X)$ when $U_n(X)$ is substituted for $x$ and $U_k(X)$ is substituted for $y$. By transfer, this formula is true in $U(^*X)$ when $^*U_n(X)$ is substituted for $x$ and $^*U_k(X)$ is substituted for $y$. Therefore $v \in ^*U_n(X)$.

(4) Use the $\Delta_0$-formula $\forall y \in x \ \forall z \in y \ (z \in x)$ with $U_n(X)$ (respectively $^*U_n(X)$) substituted for $x$ when we interpret in $U(X)$ (respectively $U(^*X)$).

By transfer, these two interpretations are equivalent, and we know that the first one is true.

(5) is an immediate consequence of (3).
The following result is an important tool for proving things about standard sets.

3.12. **Proposition** (Standard Definition Principle). Assume \( \varphi(x, y_1, \ldots, y_n) \) is a \( \Delta_0 \)-formula and \( a_1, \ldots, a_n \) are elements of \( U(X) \). Suppose further that there is a finite bound on the rank of the elements \( a \) of \( U(X) \) such that \( \varphi(a, a_1, \ldots, a_n) \) holds in \( U(X) \). Set

\[
c = \{ a \in U(X) \mid U(X) \models \varphi[a, a_1, \ldots, a_n] \} \in U(X).
\]

Then \( \{ u \in U(*) \mid u \text{ is internal and } U(*) \models \varphi[u, *a_1, \ldots, *a_n] \} \) is a standard set in \( U(*) \); indeed, we have

\[
* c = \{ u \in U(*) \mid u \text{ is internal and } U(*) \models \varphi[u, *a_1, \ldots, *a_n] \}.
\]

**Proof.** Let \( n \) be the uniform rank bound that is assumed to exist. Then \( c \subseteq U_n(X) \) and hence \( * c \subseteq * U_n(X) \) by transfer.

For each \( k \geq 0 \) we know that the following holds in \( U(X) \)

\[
\forall x \in U_k(X) \ ( \varphi(x, a_1, \ldots, a_n) \rightarrow x \in U_n(X) ).
\]

Using transfer we conclude that whenever \( u \) is internal in \( U(*) \) and \( \varphi(u, *a_1, \ldots, *a_n) \) holds in \( U(*) \), then \( u \in U_n(X) \).

Moreover, by the definition of \( c \) we have that the following holds in \( U(X) \):

\[
\forall x \in U_n(X) \ ( x \in c \leftrightarrow \varphi(x, a_1, \ldots, a_n) ).
\]

By transfer we get that the following holds in \( U(*) \):

\[
\forall x \in * U_n(X) \ ( x \in * c \leftrightarrow \varphi(x, *a_1, \ldots, *a_n) ).
\]

Combining these observations yields the desired result. \( \square \)

Combining the Standard Definition Principle with the \( \Delta_0 \)-formulas discussed in the previous section we get results about standard sets including the following:

3.13. **Proposition.** Assume \( a_1, \ldots, a_m \) are elements of \( U(X) \) and \( r, s, t, s_1, \ldots, s_n \) are sets in \( U(X) \), with \( m, n \geq 1 \) fixed.

\[(1a) \ * \emptyset = \emptyset. \]
\[(1b) \ * (s \cup t) = (*s) \cup (*t). \]
\[(1c) \ * (s \cap t) = (*s) \cap (*t). \]
\[(1d) \ * (s \setminus t) = (*s) \setminus (*t). \]
\[(2a) \ * \{ a_1, \ldots, a_m \} = \{ *a_1, \ldots, *a_m \}. \]
(2b) \( *(a_1, a_2) = (*a_1, *a_2) \).
(2c) \( *(\{a_1, a_2\} | a_1 \in s_1 \text{ and } a_2 \in s_2) = \{\{u_1, u_2\} | u_1 \in *s_1 \text{ and } u_2 \in *s_2\} \).
(2d) \( r \) is a binary relation if and only if \( *r \) is a binary relation.
(2e) If \( r \) is a binary relation, then domain of \( *r = *(\text{domain of } r) \) and range of \( *r = *(\text{range of } r) \).
(2f) \( r \) is a function if and only if \( *r \) is a function.
(2g) \( r \) is a 1-1 function if and only if \( *r \) is a 1-1 function.
(3a) \( *(a_1, \ldots, a_m) = (*a_1, \ldots, *a_m) \).
(3b) \( *(\{b_1, \ldots, b_m\} | b_1 \in s_1 \wedge \cdots \wedge b_m \in s_m) = \{\{u_1, \ldots, u_m\} | u_1 \in *s_1 \wedge \cdots \wedge u_m \in *s_m\} \).

Proof. Exercises. \(\square\)

3.14. Remark. We are now in position to explain how the present framework is a generalization of what we did in Section 1. Suppose \( X \) is a set of individuals such that \( \mathbb{R} \subseteq X \) and let \( *: U(X) \to U(*X) \) be a proper nonstandard extension. As in Section 1, let \( \mathcal{R} \) denote a structure whose underlying set is \( \mathbb{R} \). For simplicity we will assume that \( \mathcal{R} \) is entirely relational, but this is not a real restriction, since we may use familiar constructions in predicate logic to translate any formula involving functions on \( \mathbb{R} \) into an equivalent formula in which only the graphs of those functions appear. For example, if \( R \) is a binary relation and \( f, g \) are unary functions, the atomic formula \( R(x, f(g(x))) \) is logically equivalent to \( \exists u \exists v (u = g(x) \wedge v = f(u) \wedge R(x, v)) \).

We need to explain how to obtain the elementary extension \( \mathcal{R} \) of \( \mathcal{R} \) and how to translate first order formulas in the language of \( \mathcal{R} \) and \( \mathcal{R} \) into the set theoretic language of superstructures. The underlying set of \( \mathcal{R} \) is the nonstandard extension \( \mathcal{R} \) of \( \mathcal{R} \) that is given by the given mapping \( * \). For each \( n \)-ary relation \( R \) on \( \mathcal{R} \) that is a primitive in \( \mathcal{R} \), we identify \( R \) with the set of \( n \)-tuples

\( \{(r_1, \ldots, r_n) | r_1 \in \mathbb{R} \wedge \cdots \wedge r_n \in \mathbb{R} \wedge R(r_1, \ldots, r_n) \text{ holds}\} \).

With this identification, \( R \) is an element of \( U(X) \) and thus we may apply the nonstandard extension \( * \) to it. The results given above show that \( *R \) is a set of \( n \)-tuples whose coordinates come from \( \mathcal{R} \). We define \( *R(u_1, \ldots, u_n) \) to mean that the \( n \)-tuple \( (u_1, \ldots, u_n) \) is an element of \( *R \). Note that there is a \( \Delta_0 \)-formula \( \varphi(x, y_1, \ldots, y_n) \) such that \( \varphi(R, r_1, \ldots, r_n) \) holds in \( U(X) \) if and only if \( (r_1, \ldots, r_n) \in R \) and also \( \varphi(*R, u_1, \ldots, u_n) \) holds in \( U(*X) \) if and only if \( (u_1, \ldots, u_n) \in *R \).
It follows that for every formula $\sigma(x_1, \ldots, x_k)$ in the language of $R$ and $^*R$ there is a $\Delta_0$-formula $\varphi_{\sigma}(z, w_1, \ldots, w_k, x_1, \ldots, x_n)$ such that
(a) for all $r_1, \ldots, r_n \in R$ we have that $\mathcal{R} \models \sigma[r_1, \ldots, r_n]$ if and only if $\varphi_{\sigma}(R,R_1,\ldots,R_k,r_1,\ldots,r_n)$ holds in $U(X)$; and
(b) for all $u_1, \ldots, u_n \in {}^*R$ we have that $^*\mathcal{R} \models \sigma[u_1, \ldots, u_n]$ if and only if $\varphi_{\sigma}(^*\mathcal{R}, ^*R_1,\ldots,^*R_k,u_1,\ldots,u_n)$ holds in $U(^*X)$.

Here $R_1, \ldots, R_k$ are the relations on $R$ that are primitives of $R$ whose predicate symbols occur in $\sigma$.

By transfer we conclude that $^*\mathcal{R}$ is an elementary extension of $\mathcal{R}$. Moreover, it is a proper extension since we assumed $^*\mathcal{R}$ is a proper elementary extension and thus $^*\mathcal{R} \neq \mathcal{R}$.

Therefore, all concepts introduced in Section 1 (e.g., finite, infinitesimal, standard part, ...) may be used in the more general setting of nonstandard extensions of superstructures whose set of individuals contains $R$.

We conclude this section with a discussion of some basic properties of internal sets.

3.15. Proposition (Internal Definition Principle).

Suppose $\varphi(x, y_1, \ldots, y_n)$ is a $\Delta_0$-formula and $u_1, \ldots, u_n$ are internal elements of $U(^*X)$. Suppose further that there is a finite bound on the rank of the elements $u$ of $U(^*X)$ such that $\varphi(u, u_1, \ldots, u_n)$ holds in $U(^*X)$.

Then $\{u \in U(^*X) \mid u \text{ is internal and } U(^*X) \models \varphi[u, ^*u_1, \ldots, ^*u_n]\}$ is an internal set in $U(^*X)$.

Proof. Let $k$ be the finite rank bound that is assumed to exist; choose $k$ large enough so it is also a bound on the ranks of $u_1, \ldots, u_n$ in $U(^*X)$. Consider the following $\Delta_0$-formula $\sigma(y, z)$:

$$\forall x_1 \in y \ldots \forall x_n \in y \exists w \in z \forall x \in y \left( x \in w \iff \varphi(x, x_1, \ldots, x_n) \right).$$

We see that $\sigma(U_k(X), U_{k+1}(X))$ holds in $U(X)$. By transfer, we conclude that $\sigma(^*U_k(X), ^*U_{k+1}(X))$ holds in $U(^*X)$. The choice of $k$ and properties of the sets $^*U_k(X)$ and $^*U_{k+1}(X)$ that were proved above yield the desired conclusion.

Combining the Internal Definition Principle with the $\Delta_0$-formulas discussed in the previous section we get results about existence of internal sets including the following:
3.16. Proposition. Assume $u_1, \ldots, u_m$ are internal elements of $U(\ast X)$ and $A, B, C, D, A_1, \ldots, A_n$ are internal sets in $U(\ast X)$, with $m, n \geq 1$ fixed; suppose also that $f : A \to B$ is an internal function in $U(\ast X)$. The following are internal sets in $U(\ast X)$:

1. $\emptyset, A \cup B, A \cap B, A \setminus B$;
2. $\{u_1, \ldots, u_m\}, \langle u_1, u_2 \rangle, (u_1, \ldots, u_m)$;
3. $\{\langle v_1, v_2 \rangle \mid v_1 \in A_1 \land v_2 \in A_2\}$;
4. $\{(v_1, \ldots, v_n) \mid v_1 \in A_1, \ldots, v_n \in A_n\}$;
5. the set of all internal functions $g : A \to B$;
6. the set of all internal subsets of $A$;
7. the range of $f$, the restriction of $f$ to $C$, and the inverse image $f^{-1}(D)$.

Proof. Exercises. $\square$