The rest of Section 3 is devoted to properties of standard and internal sets in a nonstandard extension.

For the rest of the section we take X to be any infinite set of individuals and assume that ${}^*\colon U(X) \to U({}^*X)$ is a proper nonstandard extension.

We begin with some properties of the standard sets ${}^*U_n(X)$.

3.11. **Proposition.**
(1) ${}^*U_0(X) = {}^*X = U_0({}^*X)$*.*
(2) ${}^*U_n(X) \supseteq {}^*U_m(X)$ *for all* $n \geq m \geq 0$*.*
(3) ${}^*U_n(X)$ *is the set of all internal elements of* $U({}^*X)$ *that have rank* $\leq n$ *in the superstructure* $U({}^*X)$*, for each* $n \geq 0$*.*
(4) ${}^*U_n(X)$ *is a transitive set in* $U({}^*X)$*, for each* $n \geq 0$*.*
(5) Each element of ${}^*U_{n+1}(X) \setminus {}^*U_n(X)$ *is an internal set of rank* $n+1$ *in* $U({}^*X)$*, for each* $n \geq 0$*.*

Proof. (1) This is immediate from the definitions.
(2) This is immediate from the Transfer Principle and the fact that $U_n(X) \supseteq U_m(X)$ when $n \geq m \geq 0$.
(3) Fix $n \geq 0$. By definition, every element of ${}^*U_n(X)$ is internal. Consider the Δ_0-formula

$$\forall u \in x \; (u \text{ has rank } \leq n).$$

This formula is true in $U(X)$ when $U_n(X)$ is substituted for x. By transfer it is true in $U({}^*X)$ when ${}^*U_n(X)$ is substituted for x. This shows that every element of ${}^*U_n(X)$ has rank $\leq n$ in $U({}^*X)$.

For the converse, let v be an internal element of $U({}^*X)$ that has rank $\leq n$. By Proposition 3.8, we have $v \in {}^*U_k(X)$ for some k; by part (2) we may assume $k \geq n$. Consider the Δ_0-formula

$$\forall z \in y \; (z \text{ has rank } \leq n \;\rightarrow\; z \in x).$$

This formula is true in $U(X)$ when $U_n(X)$ is substituted for x and $U_k(X)$ is substituted for y. By transfer, this formula is true in $U({}^*X)$ when ${}^*U_n(X)$ is substituted for x and ${}^*U_k(X)$ is substituted for y. Therefore $v \in {}^*U_n(X)$.
(4) Use the Δ_0-formula $\forall y \in x \; \forall z \in y \; (z \in x)$ with $U_n(X)$ (respectively ${}^*U_n(X)$) substituted for x when we interpret in $U(X)$ (respectively $U({}^*X)$). By transfer, these two interpretations are equivalent, and we know that the first one is true.
(5) is an immediate consequence of (3). $\square$

The following result is an important tool for proving things about standard sets.

3.12. **Proposition** (Standard Definition Principle).
Suppose $\varphi(x, y_1, \dots, y_n)$ *is a* Δ_0*-formula and* $a_1, \dots, a_n$ *are elements of* $U(X)$*. Suppose further that there is a finite bound on the rank of the elements* a *of* $U(X)$ *such that* $\varphi(a, a_1, \dots, a_n)$ *holds in* $U(X)$*. Set*

$$c = \{a \in U(X) \mid \mathfrak{U}(X) \models \varphi[a, a_1, \dots, a_n]\} \in U(X).$$

Then $\{u \in U(^*X) \mid u \text{ is internal and } \mathfrak{U}(^*X) \models \varphi[u, {}^*a_1, \dots, {}^*a_n]\}$ *is a standard set in* $U(^*X)$*; indeed, we have*

$$^*c = \{u \in U(^*X) \mid u \text{ is internal and } \mathfrak{U}(^*X) \models \varphi[u, {}^*a_1, \dots, {}^*a_n]\}.$$

Proof. Let n be the uniform rank bound that is assumed to exist. Then $c \subseteq U_n(X)$ and hence $^*c \subseteq {}^*U_n(X)$ by transfer.

For each $k \geq 0$ we know that the following holds in $U(X)$

$$\forall x \in U_k(X) \ \big(\varphi(x, a_1, \dots, a_n) \rightarrow x \in U_n(X)\big).$$

Using transfer we conclude that whenever u is internal in $U(^*X)$ and $\varphi(u, {}^*a_1, \dots, {}^*a_n)$ holds in $U(^*X)$, then $u \in {}^*U_n(X)$.

Moreover, by the definition of c we have that the following holds in $U(X)$:

$$\forall x \in U_n(X) \ \big(x \in c \ \leftrightarrow \ \varphi(x, a_1, \dots, a_n)\big).$$

By transfer we get that the following holds in $U(^*X)$:

$$\forall x \in {}^*U_n(X) \ \big(x \in {}^*c \ \leftrightarrow \ \varphi(x, {}^*a_1, \dots, {}^*a_n)\big).$$

Combining these observations yields the desired result. □

Combining the Standard Definition Principle with the Δ_0-formulas discussed in the previous section we get results about standard sets including the following:

3.13. **Proposition.** *Assume* $a_1, \dots, a_m$ *are elements of* $U(X)$ *and* $r, s, t, s_1, \dots, s_n$ *are sets in* $U(X)$*, with* $m, n \geq 1$ *fixed.*
(1a) $^*\emptyset = \emptyset$.
(1b) $^*(s \cup t) = (^*s) \cup (^*t)$.
(1c) $^*(s \cap t) = (^*s) \cap (^*t)$.
(1d) $^*(s \setminus t) = (^*s) \setminus (^*t)$.
(2a) $^*\{a_1, \dots, a_m\} = \{^*a_1, \dots, {}^*a_m\}$.

(2b) $^*\langle a_1, a_2\rangle = \langle {}^*a_1, {}^*a_2\rangle$.

(2c) $^*\{\langle a_1, a_2\rangle \mid a_1 \in s_1 \text{ and } a_2 \in s_2\} = \{\langle u_1, u_2\rangle \mid u_1 \in {}^*s_1 \text{ and } u_2 \in {}^*s_2\}$.

(2d) *r is a binary relation if and only if $^*r$ is a binary relation.*

(2e) *If r is a binary relation, then* domain of $^*r = {}^*(\text{domain of } r)$ *and* range of $^*r = {}^*(\text{range of } r)$.

(2f) *r is a function if and only if $^*r$ is a function.*

(2g) *r is a 1-1 function if and only if $^*r$ is a 1-1 function.*

(3a) $^*(a_1, \ldots, a_m) = ({}^*a_1, \ldots, {}^*a_m)$.

(3b) $$^*\{(b_1, \ldots, b_m) \mid b_1 \in s_1 \wedge \cdots \wedge b_m \in s_m\} = \{(u_1, \ldots, u_m) \mid u_1 \in {}^*s_1 \wedge \cdots \wedge u_m \in {}^*s_m\}.$$

Proof. Exercises. □

3.14. **Remark.** We are now in position to explain how the present framework is a generalization of what we did in Section 1. Suppose X is a set of individuals such that $\mathbb{R} \subseteq X$ and let $^* \colon U(X) \to U({}^*X)$ be a proper nonstandard extension. As in Section 1, let $\mathcal{R}$ denote a structure whose underlying set is $\mathbb{R}$. For simplicity we will assume that $\mathcal{R}$ is entirely relational, but this is not a real restriction, since we may use familiar constructions in predicate logic to translate any formula involving functions on $\mathbb{R}$ into an equivalent formula in which only the graphs of those functions appear. For example, if R is a binary relation and f, g are unary functions, the atomic formula $R(x, f(g(x))$ is logically equivalent to $\exists u\, \exists v\, \big(u = g(x) \wedge v = f(u) \wedge R(x, v)\big)$. We need to explain how to obtain the elementary extension $^*\mathcal{R}$ of $\mathcal{R}$ and how to translate first order formulas in the language of $\mathcal{R}$ and $^*\mathcal{R}$ into the set theoretic language of superstructures. The underlying set of $^*\mathcal{R}$ is the nonstandard extension $^*\mathbb{R}$ of $\mathbb{R}$ that is given by the given mapping *. For each n-ary relation R on $^*\mathbb{R}$ that is a primitive in $\mathcal{R}$, we identify R with the set of n-tuples

$$\{(r_1, \ldots, r_n) \mid r_1 \in \mathbb{R} \wedge \cdots \wedge r_n \in \mathbb{R} \wedge R(r_1 \ldots, r_n) \text{ holds}\}.$$

With this identification, R is an element of $U(X)$ and thus we may apply the nonstandard extension * to it. The results given above show that $^*R$ is a set of n-tuples whose coordinates come from $^*\mathbb{R}$. We define $^*R(u_1, \ldots, u_n)$ to mean that the n-tuple $(u_1, \ldots, u_n)$ is an element of $^*R$. Note that there is a Δ_0-formula $\varphi(x, y_1, \ldots, y_n)$ such that $\varphi(R, r_1, \ldots, r_n)$ holds in $U(X)$ if and only if $(r_1, \ldots, r_n) \in R$ and also $\varphi({}^*R, u_1, \ldots, u_n)$ holds in $U({}^*X)$ if and only if $(u_1, \ldots, u_n) \in {}^*R$.

It follows that for every formula $\sigma(x_1,\dots,x_k)$ in the language of $\mathcal{R}$ and ${}^*\mathcal{R}$ there is a Δ_0-formula $\varphi_\sigma(z,w_1,\dots,w_k,x_1,\dots,x_n)$ such that
(a) for all $r_1,\dots,r_n \in \mathbb{R}$ we have that $\mathcal{R} \models \sigma[r_1,\dots,r_n]$ if and only if $\varphi_\sigma(\mathbb{R},R_1,\dots,R_k,r_1,\dots,r_n)$ holds in $U(X)$; and
(b) for all $u_1,\dots,u_n \in {}^*\mathbb{R}$ we have that ${}^*\mathcal{R} \models \sigma[u_1,\dots,u_n]$ if and only if $\varphi_\sigma({}^*\mathbb{R},{}^*R_1,\dots,{}^*R_k,u_1,\dots,u_n)$ holds in $U({}^*X)$.
Here $R_1,\dots,R_k$ are the relations on $\mathbb{R}$ that are primitives of $\mathcal{R}$ whose predicate symbols occur in σ.

By transfer we conclude that ${}^*\mathcal{R}$ is an elementary extension of $\mathcal{R}$. Moreover, it is a proper extension since we assumed * is a proper elementary extension and thus ${}^*\mathbb{R} \neq \mathbb{R}$.

Therefore, all concepts introduced in Section 1 (*e.g.*, finite, infinitesimal, standard part, ...) may be used in the more general setting of nonstandard extensions of superstructures whose set of individuals contains $\mathbb{R}$.

We conclude this section with a discussion of some basic properties of internal sets.

3.15. **Proposition** (Internal Definition Principle).
Suppose $\varphi(x,y_1,\dots,y_n)$ is a Δ_0-formula and $u_1,\dots,u_n$ are internal elements of $U({}^*X)$. Suppose further that there is a finite bound on the rank of the elements u of $U({}^*X)$ such that $\varphi(u,u_1,\dots,u_n)$ holds in $U({}^*X)$. Then $\{u \in U({}^*X) \mid u \text{ is internal and } \mathcal{U}({}^*X) \models \varphi[u,{}^*u_1,\dots,{}^*u_n]\}$ is an internal set in $U({}^*X)$.

Proof. Let k be the finite rank bound that is assumed to exist; choose k large enough so it is also a bound on the ranks of $u_1,\dots,u_n$ in $U({}^*X)$. Consider the following Δ_0-formula $\sigma(y,z)$:

$$\forall x_1 \in y \ \dots\ \forall x_n \in y\ \exists w \in z\ \forall x \in y\ \big(x \in w \ \leftrightarrow\ \varphi(x,x_1,\dots,x_n)\big).$$

We see that $\sigma(U_k(X),U_{k+1}(X))$ holds in $U(X)$. By transfer, we conclude that $\sigma({}^*U_k(X),{}^*U_{k+1}(X))$ holds in $U({}^*X)$. The choice of k and properties of the sets ${}^*U_k(X)$ and ${}^*U_{k+1}(X)$ that were proved above yield the desired conclusion. $\square$

Combining the Internal Definition Principle with the Δ_0-formulas discussed in the previous section we get results about existence of internal sets including the following:

3.16. **Proposition.** *Assume* $u_1, \dots, u_m$ *are internal elements of* $U({}^*X)$ *and* $A, B, C, D, A_1, \dots, A_n$ *are internal sets in* $U({}^*X)$*, with* $m, n \geq 1$ *fixed; suppose also that* $f\colon A \to B$ *is an internal function in* $U({}^*X)$*. The following are internal sets in* $U({}^*X)$*:*

(1) $\emptyset, A \cup B, A \cap B, A \setminus B$*;*

(2) $\{u_1, \dots, u_m\}, \langle u_1, u_2 \rangle, (u_1, \dots, u_m)$*;*

(3) $\{\langle v_1, v_2 \rangle \mid v_1 \in A_1 \wedge v_2 \in A_2\}$*;*

(4) $\{(v_1, \dots, v_n) \mid v_1 \in A_1, \dots, v_n \in A_n\}$*;*

(5) the set of all internal functions $g\colon A \to B$*;*

(6) the set of all internal subsets of A*;*

(7) the range of f*, the restriction of* f *to* C*, and the inverse image* $f^{-1}(D)$*.*

Proof. Exercises. □