4. Basic Concepts

In this section we take \( X \) to be any infinite set of individuals that contains \( \mathbb{R} \) as a subset and we assume that \( *: U(X) \rightarrow U(*X) \) is a proper nonstandard extension.

The purpose of this section is to introduce three important concepts that are characteristic of arguments using nonstandard analysis: \textit{overspill} and \textit{underspill} (consequences of certain sets in \( U(*X) \) not being internal); \textit{hyperfinite sets} and \textit{hyperfinite sums} (combinatorics of hyperfinite objects in \( U(*X) \)); and \textit{saturation}.

**Overspill and underspill**

4.1. \textbf{Lemma.} The sets \( \mathbb{N} \), \( \mu(0) \), and \( \text{fin}(\mathbb{R}) \) are external in \( U(*X) \).

\textit{Proof.} Every bounded nonempty subset of \( \mathbb{N} \) has a maximum element. By transfer we conclude that every bounded nonempty internal subset of \( *\mathbb{N} \) has a maximum element. Since \( \mathbb{N} \) is a subset of \( *\mathbb{N} \) that is bounded above (by any infinite element of \( *\mathbb{N} \)) but that has no maximum element, it follows that \( \mathbb{N} \) is external.

Every bounded nonempty subset of \( \mathbb{R} \) has a least upper bound. By transfer we conclude that every bounded nonempty internal subset of \( *\mathbb{R} \) has a least upper bound. Since \( \mu(0) \) is a bounded nonempty subset of \( *\mathbb{R} \) that has no least upper bound, it follows that \( \mu(0) \) is external.

If \( \text{fin}(\mathbb{R}) \) were internal, so would \( \mathbb{N} = \text{fin}(\mathbb{R}) \cap *\mathbb{N} \) be internal. Since \( \mathbb{N} \) is external, it follows that \( \text{fin}(\mathbb{R}) \) is also external. \( \square \)

4.2. \textbf{Proposition.} (Overspill and Underspill Principles) Let \( A \) be an internal set in \( U(*X) \).

(1) (For \( *\mathbb{N} \)) \( A \) contains arbitrarily large elements of \( \mathbb{N} \) if and only if \( A \) contains arbitrarily small infinite elements of \( *\mathbb{N} \).

(2) (For \( \mu(0) \)) \( A \) contains arbitrarily large positive infinitesimals from \( *\mathbb{R} \) if and only if \( A \) contains finite elements from \( *\mathbb{R} \) whose standard parts are arbitrarily small \( > 0 \).

(3) (For \( \text{fin}(\mathbb{R}) \)) \( A \) contains arbitrarily large positive finite elements of \( *\mathbb{R} \) if and only if \( A \) contains arbitrarily small positive infinite elements of \( *\mathbb{R} \).
Proof. (1) We use the fact that $\mathbb{N}$ is external, and argue by contradiction in both directions.

($\Rightarrow$) Suppose $A$ contains arbitrarily large elements of $\mathbb{N}$ and there exists an infinite element $H$ of $^*\mathbb{N}$ such that every $a \in A \cap ^*\mathbb{N}$ satisfying $a < H$ is in $\mathbb{N}$. We conclude

$$\mathbb{N} = \{ b \in ^*\mathbb{N} \mid \exists a \in A (a \in ^*\mathbb{N} \land b < a \land a < H) \}.$$ 

Using the Internal Definition Principle we have that $\mathbb{N}$ is internal, which is a contradiction.

($\Leftarrow$) Suppose $A$ contains arbitrarily small infinite elements of $^*\mathbb{N}$ and there exists $k \in \mathbb{N}$ such that every element of $A \cap \mathbb{N}$ is $\leq k$. We conclude

$$\mathbb{N} = \{ b \in \mathbb{N} \mid \forall a \in A ((a \in ^*\mathbb{N} \land k < a) \rightarrow b < a) \}.$$ 

Again using the Internal Definition Principle we have proved the false statement that $\mathbb{N}$ is internal.

(2) and (3) Exercises. The proofs are similar to the proof of (1). □

4.3. Remark. Let $A$ be an internal set in $U(^*X)$. There are many variants of the principles stated in the previous result that are easily derived from what is stated there. For example:

(4) If $A$ contains $\{ n \in \mathbb{N} \mid k \leq n \}$ for some $k \in \mathbb{N}$, then there exists an infinite element $H$ of $^*\mathbb{N}$ such that $A \supseteq \{ c \in ^*\mathbb{N} \mid k \leq c \leq H \}$.

(5) Let $r \in \mathbb{R}$. If $A$ contains $\mu(r)$, then there exists $\delta > 0$ in $\mathbb{R}$ such that $A \supseteq ^*(r - \delta, r + \delta)$.

Hyperfinite sets

A set $x$ is finite if there exists $n \in \mathbb{N}$ and a 1-1 function $f$ from $\{ m \in \mathbb{N} \mid m < n \}$ onto $x$. Moreover, in this case $n$ is unique and it is called the cardinality or size of $x$. When $x$ is a finite set and $n$ is the cardinality of $x$, we write $n = \operatorname{card}(x)$ or $n = |x|$.

4.4. Definition. Let $A$ be a set in $U(^*X)$. We say $A$ is hyperfinite if there exists $H \in ^*\mathbb{N}$ and an internal 1-1 function from $\{ u \in ^*\mathbb{N} \mid u < H \}$ onto $A$.

Note that a hyperfinite set is necessarily internal, since it is the range of an internal function.
4.5. **Remark.** If we fix \( n \geq 1 \), then conditions such as the following are expressible in \( U(^*X) \) by \( \Delta_0 \)-formulas in which a name for \( ^*U_{n+2}(X) \) occurs.

1. \( A \) is a hyperfinite set of rank \( \leq n \).
2. \( A \) is a hyperfinite set of rank \( \leq n \), \( H \in ^*\mathbb{N} \), and \( H = |A| \).

The reason that \( ^*U_{n+2}(X) \) needs to be mentioned in the formulas is that the internal function witnessing that \( A \) is hyperfinite will be an internal set of rank \( \leq n + 2 \) when \( A \) has rank \( \leq n \), and we need to express the existence of such a function using a bounded quantifier.

4.6. **Lemma.** If \( A \) is a hyperfinite set in \( U(^*X) \), then the \( H \in ^*\mathbb{N} \) such that there exists an internal 1-1 function from \( \{ u \in ^*\mathbb{N} \mid u < H \} \) onto \( A \) is unique.

*Proof.* Use the \( \Delta_0 \)-formulas discussed above and in Section 2, and the Transfer Principle. \( \square \)

4.7. **Definition.** If \( A \) is a hyperfinite set in \( U(^*X) \) and \( H \) is the unique element of \( ^*\mathbb{N} \) such that there exists an internal 1-1 function from \( \{ u \in ^*\mathbb{N} \mid u < H \} \) onto \( A \), then \( H \) is called the (internal) cardinality of \( A \) and we write \( H = \text{card}(A) \) or \( H = |A| \).

An important example of a hyperfinite set is

\[
\{ \frac{m}{H} \mid m \in ^*\mathbb{N} \land m \leq H \}
\]

where \( H \) is an infinite element of \( ^*\mathbb{N} \). The internal cardinality of this set is \( H + 1 \) and the internal 1-1 function witnessing that the set is hyperfinite takes \( m \) to \( m/H \) for \( m \in \{0, 1, \ldots, H\} \).

4.8. **Notation.** For any set \( A \) we let \( \mathcal{P}(A) \) denote the set of all subsets of \( A \), and \( \mathcal{P}_f(A) \) will denote the set of all finite subsets of \( A \).

4.9. **Proposition.**

1. If \( a \) is a set in \( U(X) \), then \( ^*\mathcal{P}(a) \) (respectively, \( ^*\mathcal{P}_f(a) \)) is the set of all internal subsets of \( ^*a \) (respectively, the set of all hyperfinite subsets of \( ^*a \)).
2. If \( A \) is an internal set in \( U(^*X) \), then the set of all hyperfinite subsets of \( A \) is internal.

*Proof.* (1) Apply the Standard Definition Principle to the \( \Delta_0 \)-formula \( \neg I(x) \land \forall y \in x \ (y \in a) \). Note that if \( A \) has rank \( n \), then every \( x \) satisfying this formula has rank at most \( n \), so this Principle applies.
(2) Suppose $A$ has rank $n$. Consider the $\Delta_0$-formula $\varphi(x, y, u, v, w)$ such that $\varphi(x, y, N, <, U_{n+2}(X))$ formalizes the condition “$x \subseteq y$ and there exists a 1-1 function $f \in U_{n+2}(X)$ whose range is $x$ and whose domain is $\{0, 1, \ldots, H\}$ for some $H \in \mathbb{N}$”. We apply the Internal Definition Principle to the formula $\varphi(x, A, *N, *, *U_{n+2}(X))$, whose meaning is: “$x \subseteq y$ and there exists a 1-1 function $f \in *U_{n+2}(X)$ whose range is $x$ and whose domain is $\{0, 1, \ldots, H\}$ for some $H \in \mathbb{N}$”. Such functions are necessarily internal, which implies that any $x$ satisfying this formula will be a hyperfinite subset of $A$. Moreover, if $x$ is a hyperfinite subset of $A$, then the internal function witnessing that $x$ is hyperfinite will indeed be an element of $*U_{n+2}(X)$. Thus $\{x \in U(*X) \mid x$ is hyperfinite and $x \subseteq A\}$ is equal to $\{x \in U(*X) \mid x$ is internal and $\varphi(x, A, *N, *, *U_{n+2}(X))\}$ and this set is internal by the Internal Definition Principle.

4.10. Notation. For any internal set $A$ in $U(*X)$ we let $*P(A)$ denote the set of all internal subsets of $A$, and $*P_f(A)$ will denote the set of all hyperfinite subsets of $A$. If $A = *a$ for some set $a \in U(X)$, then we have two notations for the same sets: $*P(*a) = *(P(a))$ and also $*P_f(*a) = *(P_f(a))$.

Next we give a list of some basic properties of hyperfinite sets.

4.11. Proposition (Properties of hyperfinite sets). Let $A, B$ be internal sets and $f : A \to B$ an internal function.

(1) Assume $B$ is hyperfinite and $f$ is 1-1. Then $A$ is hyperfinite and $|A| \leq |B|$; moreover, $|A| = |B|$ if and only if $f$ is surjective. (In particular these hold when $B$ is hyperfinite and $A$ is an internal subset of $B$.)

(2) Assume $A$ is hyperfinite and $f$ is surjective. Then $B$ is hyperfinite and $|B| \leq |A|$; moreover, $|A| = |B|$ if and only if $f$ is injective.

(3) Assume $A, B$ are both hyperfinite. Then $A \cup B$ is hyperfinite and $|A \cup B| + |A \cap B| = |A| + |B|$.

(4) Assume $A, B$ are both hyperfinite. Then $A \times B$ is hyperfinite and $|A \times B| = |A| \cdot |B|$.

(5) Assume $A$ is hyperfinite. Then $*P(A)$ is hyperfinite and $|*P(A)| = 2^{|A|}$. (Here $2^{|A|}$ is taken to be $*E(|A|)$ where $E : \mathbb{N} \to \mathbb{N}$ is the function given by $E(n) = 2^n$ for all $n \in \mathbb{N}$.)

Proof. Exercises. Results like these are easily proved by formulating the corresponding properties of finite sets in $U(X)$ using $\Delta_0$-formulas (in which $U_k(X)$ occurs for suitable $k$), applying the Transfer Principle, and then
interpreting the meaning of the resulting formula in the nonstandard extension.

Hyperfinite sums

Suppose \( A \) is a hyperfinite set in \( U(*X) \) and \( f \) is an internal function whose domain contains \( A \) and whose range is in \( *\mathbb{R} \). We want to give a meaning to the expression “the sum of \( f(x) \) as \( x \) ranges over \( A \)”, which we will denote by

\[
\sum_{x \in A} f(x).
\]

Fix \( n \geq 1 \). Let \( \sum \) denote the function in \( U(X) \) whose domain is the set of all \((a, f)\) such that \( a \) is a finite set of rank \( \leq n \) and \( f \) is a function of rank \( \leq n + 2 \) whose domain contains \( a \) and whose range is contained in \( \mathbb{R} \), and whose value at such an \((a, f)\) is given by

\[
\sum (a, f) = \sum_{x \in a} f(x).
\]

It is easy to see that the domain of the nonstandard extension \(*\sum\) is the set of all \((A, f)\) such that \( A \) is a hyperfinite set of rank \( \leq n \) and \( f \) is an internal function of rank \( \leq n+2 \) whose domain contains \( A \) and whose range is contained in \(*\mathbb{R} \). For any such \((A, f)\) we make the definition

\[
\sum_{x \in A} f(x) = *\sum (A, f).
\]

It is easy to check, using the Transfer Principle, that this definition does not depend on the rank bound \( n \).

4.12. Proposition (Properties of hyperfinite sums). Suppose \( A, B \) are hyperfinite sets and \( f, g \) are internal functions whose domains contain \( A \) and \( B \) and whose ranges are contained in \(*\mathbb{R} \). Then

1. For any \( u \in (*\mathbb{R}, \sum_{x \in A} uf(x) = u \sum_{x \in A} f(x).) \)
2. \( \sum_{x \in A} (f(x) + g(x)) = \sum_{x \in A} f(x) + \sum_{x \in A} g(x).) \)
3. If \( A \cap B = \emptyset \), then \( \sum_{x \in A \cup B} f(x) = \sum_{x \in A} f(x) + \sum_{x \in B} f(x).) \)
4. If \( f(x) \leq g(x) \) for all \( x \in A \), then \( \sum_{x \in A} f(x) \leq \sum_{x \in A} g(x).) \)

Proof. Exercises. Bound the ranks of \( A, B, f, g \) by \( n \) and express properties of the function \( \sum(a, f) \) by appropriate \( \Delta_0 \)-formulas in which \( U_{n+2}(X) \) occurs. Check that the Transfer Principle yields properties of the nonstandard extension \(*\sum\) that correspond to the items in the Proposition. \( \square \)
Saturation

Fix an uncountable cardinal number $\kappa$.

Recall that a collection of sets $\mathcal{F}$ is said to have the finite intersection property (FIP) if every finite subcollection of $\mathcal{F}$ has nonempty intersection.

4.13. Definition. The nonstandard extension $*: U(X) \rightarrow U(\ast X)$ is said to be $\kappa$-saturated if, whenever $I$ is a set of cardinality $< \kappa$ and $\{A_i \mid i \in I\}$ is a collection of internal sets with the finite intersection property, then $\bigcap\{A_i \mid i \in I\} \neq \emptyset$.

4.14. Theorem (Existence of $\kappa$-saturated nonstandard extensions). For every infinite set of individuals $X$ there exists a $\kappa$-saturated nonstandard extension $*: U(X) \rightarrow U(\ast X)$. Every such nonstandard extension is proper.

Proof. Using basic model theory we may obtain a $\kappa$-saturated $L$-structure $M$ and an elementary embedding $F$ of $(U(X), \in, X)$ into $M$. Apply the Mostowski collapsing construction as in the proof of Theorem 3.4 to obtain a nonstandard extension $*: U(X) \rightarrow U(\ast X)$ with $\ast X = I^M$. We will show that this nonstandard extension is $\kappa$-saturated. Recall that this construction uses the substructure $M_f$ of all strongly well-founded elements of $M$ and an embedding $G$ of $M_f$ into $U(\ast X)$. The range of $G$ is exactly the transitive substructure of $U(\ast X)$ consisting of the internal elements.

Suppose $I$ is a set of cardinality less than $\kappa$ and $\{A_i \mid i \in I\}$ is a collection of internal sets in $U(\ast X)$ that has the finite intersection property. For each $i \in I$, let $m_i \in M_f$ be such that $G(m_i) = A_i$. We consider the $L(M)$-formulas $xEm_i$ for $i \in I$. The fact that $\{A_i \mid i \in I\}$ has the FIP ensures that $\{xEm_i \mid i \in I\}$ is finitely satisfiable in $M$. Since $M$ is $\kappa$-saturated and $\text{card}(I) < \kappa$, there exists $m \in M$ such that $mE^Mm_i$ holds for all $i \in I$. It follows that $m$ is strongly well founded, and we see that $G(m)$ is an element of the intersection of the collection $\{A_i \mid i \in I\}$. Therefore we have shown that the nonstandard extension $*: U(X) \rightarrow U(\ast X)$ is $\kappa$-saturated.

Finally we show that any $\kappa$-saturated extension is proper. Let $C$ be any countable, infinite subset of $X$ and consider the internal sets $A_c = \{x \in \ast X \mid x \neq c\}$ for $c \in C$. Since $C$ is infinite, this collection of sets has the FIP. Since $C$ is countable and the nonstandard extension is $\kappa$-saturated (with $\kappa > \omega$), we conclude that $\bigcap\{A_c \mid c \in C\}$ is nonempty. That is, there exists
$u \in \ast C$ that is distinct from $c$ for all $c \in C$, and hence the nonstandard extension is proper. \hfill \Box

4.15. Proposition (Comprehension principle).

(Assume $\kappa$-saturation.) Let $A, B$ be internal sets in $U(\ast X)$ and let $S$ be a subset of $A$ with $\text{card}(S) < \kappa$. For every function $\alpha : S \to B$, there exists an internal function $f : A \to B$ whose restriction to $S$ is $\alpha$.

Proof. Assume $A, B, S, \alpha$ are as in the hypotheses. Let $a_0$ be a fixed element of $A$. For each $x \in S$ let

$$A_x = \{ g \mid g : A \to B \text{ is internal and } g(x) = \alpha(x) \}.$$ 

Each $A_x$ is an internal set. Moreover, the collection of sets $\{ A_x \mid x \in S \}$ has the finite intersection property. Indeed, for each finite set $F \subseteq S$ the function $g : A \to B$ defined by

$$g(a) = \begin{cases} \alpha(a) & a \in F \\ \alpha(a_0) & \text{otherwise} \end{cases}$$

is internal and is an element of every $A_x$ for $x \in F$. Using $\kappa$-saturation we conclude that $\bigcap_{x \in S} A_x$ contains some element $g$, which is necessarily an internal function $g : A \to B$ that extends $\alpha$. \hfill \Box

The next result formulates the $\kappa$-saturation condition in a way that will be useful in many situations.

4.16. Proposition. (Assume $\kappa$-saturation.) Let $I, J$ be index sets, each of cardinality $< \kappa$. For each $i \in I$ and $j \in J$ let $A_i$ and $B_j$ be internal sets, and assume that all of the sets in $\{ A_i \mid i \in I \} \cup \{ B_j \mid j \in J \}$ have rank $\leq n$ in $U(\ast X)$. Then the following two conditions are equivalent:

(1) $\bigcup \{ A_i \mid i \in I \} \supseteq \bigcap \{ B_j \mid j \in J \}$.

(2) There exist finite sets $I_0 \subseteq I$ and $J_0 \subseteq J$ such that

$$\bigcup \{ A_i \mid i \in I_0 \} \supseteq \bigcap \{ B_j \mid j \in J_0 \}.$$ 

Proof. Let the families $(A_i \mid i \in I)$ and $(B_j \mid j \in J)$ be as in the hypotheses. Note that every $A_i$ and every $B_j$ is a subset of $\ast U_n(X)$. The equivalence between (1) and (2) follows immediately from an application of $\kappa$-saturation to the family of internal sets consisting of $\ast U_n(X) \setminus A_i$ for $i \in I$ and $B_j$ for $j \in J$. Note that (1) holds iff the intersection of these sets is empty. \hfill \Box
4.17. Example. (Assume $\omega_1$-saturation.) Let $A$ be a hyperfinite set in $U(\ast X)$ and let $\mathcal{A}$ be the boolean algebra of all internal subsets of $A$. For each $B \in \mathcal{A}$ define
\[ \mu(B) = \text{st}\left(\frac{|B|}{|A|}\right). \]
Note that this standard part is well defined because $0 \leq |B| \leq |A|$. Using properties of the standard part and of internal cardinalities of hyperfinite sets, it is easy to show that $\mu$ is a finitely additive probability measure on $\mathcal{A}$.

Moreover, because of $\omega_1$-saturation, we have that $\mu$ satisfies the hypotheses of the Carathéodory Extension Theorem on $\mathcal{A}$. That is if $\left\{A_n \mid n \in \mathbb{N}\right\}$ is a descending sequence in $\mathcal{A}$ and $A \in \mathcal{A}$ is equal to $\bigcap\{A_n \mid n \in \mathbb{N}\}$, then $\mu(A_n)$ converges to $\mu(A)$ as $n \to \infty$. Indeed, this is true in a very strong way: using Proposition 4.16, there exists $k \in \mathbb{N}$ such that $A_n = A$ for all $n \geq k$ in $\mathbb{N}$.

Therefore we conclude that $\mu$ extends to a $\sigma$-additive probability measure defined on the $\sigma$-algebra of subsets of $A$ generated by $A$. This is an example of a Loeb measure, which play an important role in many applications of nonstandard analysis. These measures will be extensively studied later in the course.

4.18. Exercises. (Assume $\kappa$-saturation, $\kappa \geq \omega_1$.)
1. Suppose $S, T$ are subsets of $\ast \mathbb{R}$ with $S < T$. (That is, for any $x \in S$ and $y \in T$, we have $x < y$.) Suppose further that $S$ and $T$ have cardinality $< \kappa$. Then there exists $a \in \ast \mathbb{R}$ such that $S < a < T$. In particular, if each element of $T$ is positive infinite, then there exists a positive infinite $a$ with $a < T$. (Take $S = \mathbb{N}$).
2. If $A$ is an infinite internal set, then $A$ has cardinality $\geq \kappa$.
3. Suppose $\left\{A_n \mid n \in \mathbb{N}\right\}$ is a sequence of distinct internal sets. If $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$, then $\bigcup\{A_n \mid n \in \mathbb{N}\}$ is external. Likewise, if $A_0 \supset A_1 \supset \cdots \supset A_n \supset \cdots$, then $\bigcap\{A_n \mid n \in \mathbb{N}\}$ is external.