4. Basic Concepts

In this section we take X to be any infinite set of individuals that contains $\mathbb{R}$ as a subset and we assume that $^*\colon U(X) \to U(^*X)$ is a proper nonstandard extension.

The purpose of this section is to introduce three important concepts that are characteristic of arguments using nonstandard analysis: *overspill* and *underspill* (consequences of certain sets in $U(^*X)$ not being internal); *hyperfinite sets* and *hyperfinite sums* (combinatorics of hyperfinite objects in $U(^*X)$); and *saturation*.

Overspill and underspill

4.1. **Lemma.** *The sets* $\mathbb{N}$, $\mu(0)$, *and* $\text{fin}(^*R)$ *are external in* $U(^*X)$.

Proof. Every bounded nonempty subset of $\mathbb{N}$ has a maximum element. By transfer we conclude that every bounded nonempty *internal* subset of $^*N$ has a maximum element. Since $\mathbb{N}$ is a subset of $^*\mathbb{N}$ that is bounded above (by any infinite element of $^*\mathbb{N}$) but that has no maximum element, it follows that $\mathbb{N}$ is external.

Every bounded nonempty subset of $\mathbb{R}$ has a least upper bound. By transfer we conclude that every bounded nonempty internal subset of $^*\mathbb{R}$ has a least upper bound. Since $\mu(0)$ is a bounded nonempty subset of $^*\mathbb{R}$ that has no least upper bound, it follows that $\mu(0)$ is external.

If $\text{fin}(^*\mathbb{R})$ were internal, so would $\mathbb{N} = \text{fin}(^*\mathbb{R}) \cap {}^*N$ be internal. Since $\mathbb{N}$ is external, it follows that $\text{fin}(^*\mathbb{R})$ is also external. $\square$

4.2. **Proposition.** *(Overspill and Underspill Principles) Let A be an internal set in $U(^*X)$.*
(1) (For $^*\mathbb{N}$) A contains arbitrarily large elements of $\mathbb{N}$ if and only if A contains arbitrarily small infinite elements of $^*\mathbb{N}$.
(2) (For $\mu(0)$) A contains arbitrarily large positive infinitesimals from $^*\mathbb{R}$ if and only if A contains finite elements from $^*\mathbb{R}$ whose standard parts are arbitrarily small > 0.
(3) (For $\text{fin}(^*\mathbb{R})$) A contains arbitrarily large positive finite elements of $^*\mathbb{R}$ if and only if A contains arbitrarily small positive infinite elements of $^*R$.

Proof. (1) We use the fact that $\mathbb{N}$ is external, and argue by contradiction in both directions.

($\Rightarrow$) Suppose A contains arbitrarily large elements of $\mathbb{N}$ and there exists an infinite element H of ${}^*\mathbb{N}$ such that every $a \in A \cap {}^*\mathbb{N}$ satisfying $a < H$ is in $\mathbb{N}$. We conclude

$$\mathbb{N} = \{b \in {}^*\mathbb{N} \mid \exists a \in A\ (a \in {}^*\mathbb{N} \wedge b < a \wedge a < H)\}.$$

Using the Internal Definition Principle we have that $\mathbb{N}$ is internal, which is a contradiction.

($\Leftarrow$) Suppose A contains arbitrarily small infinite elements of ${}^*\mathbb{N}$ and there exists $k \in \mathbb{N}$ such that every element of $A \cap \mathbb{N}$ is $\leq k$. We conclude

$$\mathbb{N} = \{b \in {}^*\mathbb{N} \mid \forall a \in A\ \big((a \in {}^*\mathbb{N} \wedge k < a) \rightarrow b < a\big).\}$$

Again using the Internal Definition Principle we have proved the false statement that $\mathbb{N}$ is internal.

(2) and (3) Exercises. The proofs are similar to the proof of (1). □

4.3. **Remark.** Let A be an internal set in $U({}^*X)$. There are many variants of the principles stated in the previous result that are easily derived from what is stated there. For example:
(4) If A contains $\{n \in \mathbb{N} \mid k \leq n\}$ for some $k \in \mathbb{N}$, then there exists an infinite element H of ${}^*\mathbb{N}$ such that $A \supseteq \{c \in {}^*\mathbb{N} \mid k \leq c \leq H\}$.
(5) Let $r \in \mathbb{R}$. If A contains $\mu(r)$, then there exists $\delta > 0$ in $\mathbb{R}$ such that $A \supseteq {}^*(r - \delta, r + \delta)$.

Hyperfinite sets

A set x is finite if there exists $n \in \mathbb{N}$ and a 1-1 function f from $\{m \in \mathbb{N} \mid m < n\}$ onto x. Moreover, in this case n is unique and it is called the *cardinality* or *size* of x. When x is a finite set and n is the cardinality of x, we write $n = \text{card}(x)$ or $n = |x|$.

4.4. **Definition.** Let A be a set in $U({}^*X)$. We say A is *hyperfinite* if there exists $H \in {}^*\mathbb{N}$ and an internal 1-1 function from $\{u \in {}^*\mathbb{N} \mid u < H\}$ onto A.

Note that a hyperfinite set is necessarily internal, since it is the range of an internal function.

4.5. **Remark.** If we fix $n \geq 1$, then conditions such as the following are expressible in $U(^*X)$ by Δ_0-formulas in which a name for $^*U_{n+2}(X)$ occurs.
(1) A is a hyperfinite set of rank $\leq n$.
(2) A is a hyperfinite set of rank $\leq n$, $H \in {}^*\mathbb{N}$, and $H = |A|$.
The reason that $^*U_{n+2}(X)$ needs to be mentioned in the formulas is that the internal function witnessing that A is hyperfinite will be an internal set of rank $\leq n+2$ when A has rank $\leq n$, and we need to express the existence of such a function using a bounded quantifier.

4.6. **Lemma.** *If A is a hyperfinite set in $U(^*X)$, then the $H \in {}^*\mathbb{N}$ such that there exists an internal 1-1 function from $\{u \in {}^*\mathbb{N} \mid u < H\}$ onto A is* unique.

Proof. Use the Δ_0-formulas discussed above and in Section 2, and the Transfer Principle. ☐

4.7. **Definition.** If A is a hyperfinite set in $U(^*X)$ and H is the unique element of $^*\mathbb{N}$ such that there exists an internal 1-1 function from $\{u \in {}^*\mathbb{N} \mid u < H\}$ onto A, then H is called the *(internal) cardinality* of A and we write $H = \text{card}(A)$ or $H = |A|$.

An important example of a hyperfinite set is

$$\{\frac{m}{H} \mid m \in {}^*\mathbb{N} \wedge m \leq H\}$$

where H is an infinite element of $^*\mathbb{N}$. The internal cardinality of this set is $H+1$ and the internal 1-1 function witnessing that the set is hyperfinite takes m to m/H for $m \in \{0, 1, \ldots, H\}$.

4.8. **Notation.** For any set A we let $\mathcal{P}(A)$ denote the set of all subsets of A, and $\mathcal{P}_f(A)$ will denote the set of all *finite* subsets of A.

4.9. **Proposition.**
(1) If a is a set in $U(X)$, then $^*\mathcal{P}(a)$ (respectively, $^*\mathcal{P}_f(a)$) is the set of all internal subsets of $^*a$ (respectively, the set of all hyperfinite subsets of $^*a$).
(2) If A is an internal set in $U(^*X)$, then the set of all hyperfinite subsets of A is internal.

Proof. (1) Apply the Standard Definition Principle to the Δ_0-formula $\neg I(x) \wedge \forall y \in x\ (y \in a)$. Note that if A has rank n, then every x satisfying this formula has rank at most n, so this Principle applies.

(2) Suppose A has rank n. Consider the Δ_0-formula $\varphi(x, y, u, v, w)$ such that $\varphi(x, y, \mathbb{N}, <, U_{n+2}(X))$ formalizes the condition "$x \subseteq y$ and there exists a 1-1 function $f \in U_{n+2}(X)$ whose range is x and whose domain is $\{0, 1, \ldots, H\}$ for some $H \in \mathbb{N}$". We apply the Internal Definition Principle to the formula $\varphi(x, A, {}^*\mathbb{N}, {}^*<, {}^*U_{n+2}(X))$, whose meaning is: "$x \subseteq y$ and there exists a 1-1 function $f \in {}^*U_{n+2}(X)$ whose range is x and whose domain is $\{0, 1, \ldots, H\}$ for some $H \in \mathbb{N}$". Such functions are necessarily internal, which implies that any x satisfying this formula will be a hyperfinite subset of A. Moreover, if x is a hyperfinite subset of A, then the internal function witnessing that x is hyperfinite will indeed be an element of ${}^*U_{n+2}(X)$. Thus $\{x \in U({}^*X) \mid x \text{ is hyperfinite and } x \subseteq A\}$ is equal to $\{x \in U({}^*X) \mid x \text{ is internal and } \varphi(x, A, {}^*\mathbb{N}, {}^*<, {}^*U_{n+2}(X))\}$ and this set is internal by the Internal Definition Principle. □

4.10. **Notation.** For any internal set A in $U({}^*X)$ we let ${}^*\mathcal{P}(A)$ denote the set of all internal subsets of A, and ${}^*\mathcal{P}_f(A)$ will denote the set of all hyperfinite subsets of A. If $A = {}^*a$ for some set $a \in U(X)$, then we have two notations for the same sets: ${}^*\mathcal{P}({}^*a) = {}^*\big(\mathcal{P}(a)\big)$ and also ${}^*\mathcal{P}_f({}^*a) = {}^*\big(\mathcal{P}_f(a)\big)$.

Next we give a list of some basic properties of hyperfinite sets.

4.11. **Proposition** (Properties of hyperfinite sets)**.** *Let A, B be internal sets and $f\colon A \to B$ an internal function.*
(1) Assume B is hyperfinite and f is 1-1. Then A is hyperfinite and $|A| \leq |B|$; moreover, $|A| = |B|$ if and only if f is surjective. (In particular these hold when B is hyperfinite and A is an internal subset of B.)
(2) Assume A is hyperfinite and f is surjective. Then B is hyperfinite and $|B| \leq |A|$; moreover, $|A| = |B|$ if and only if f is injective.
(3) Assume A, B are both hyperfinite. Then $A \cup B$ is hyperfinite and
$$|A \cup B| + |A \cap B| = |A| + |B|.$$
(4) Assume A, B are both hyperfinite. Then $A \times B$ is hyperfinite and
$$|A \times B| = |A| \cdot |B|.$$
(5) Assume A is hyperfinite. Then ${}^*\mathcal{P}(A)$ is hyperfinite and $|{}^*\mathcal{P}(A)| = 2^{|A|}$. (Here $2^{|A|}$ is taken to be ${}^*E(|A|)$ where $E\colon \mathbb{N} \to \mathbb{N}$ is the function given by $E(n) = 2^n$ for all $n \in \mathbb{N}$.)

Proof. Exercises. Results like these are easily proved by formulating the corresponding properties of finite sets in $U(X)$ using Δ_0-formulas (in which $U_k(X)$ occurs for suitable k), applying the Transfer Principle, and then interpreting the meaning of the resulting formula in the nonstandard extension. □

Hyperfinite sums

Suppose A is a hyperfinite set in $U({}^*X)$ and f is an internal function whose domain contains A and whose range is in ${}^*\mathbb{R}$. We want to give a meaning to the expression "*the sum of* $f(x)$ *as* x *ranges over* A", which we will denote by

$$\sum_{x\in A} f(x).$$

Fix $n \geq 1$. Let $\sum$ denote the function in $U(X)$ whose domain is the set of all (a, f) such that a is a finite set of rank $\leq n$ and f is a function of rank $\leq n+2$ whose domain contains a and whose range is contained in $\mathbb{R}$, and whose value at such an (a, f) is given by

$$\sum(a, f) = \sum_{x\in a} f(x).$$

It is easy to see that the domain of the nonstandard extension ${}^*\sum$ is the set of all (A, f) such that A is a hyperfinite set of rank $\leq n$ and f is an internal function of rank $\leq n+2$ whose domain contains A and whose range is contained in ${}^*\mathbb{R}$. For any such (A, f) we make the definition

$$\sum_{x\in A} f(x) = {}^*\sum(A, f).$$

It is easy to check, using the Transfer Principle, that this definition does not depend on the rank bound n.

4.12. **Proposition** (Properties of hyperfinite sums)**.** *Suppose A, B are hyperfinite sets and f, g are internal functions whose domains contain A and B and whose ranges are contained in ${}^*\mathbb{R}$. Then*
(1) For any $u \in {}^*\mathbb{R}$, $\sum_{x\in A} uf(x) = u\sum_{x\in A} f(x)$.
(2) $\sum_{x\in A} \big(f(x) + g(x)\big) = \sum_{x\in A} f(x) + \sum_{x\in A} g(x)$.
(3) If $A \cap B = \emptyset$, then $\sum_{x\in A\cup B} f(x) = \sum_{x\in A} f(x) + \sum_{x\in B} f(x)$.
(4) If $f(x) \leq g(x)$ for all $x \in A$, then $\sum_{x\in A} f(x) \leq \sum_{x\in A} g(x)$.

Proof. Exercises. Bound the ranks of A, B, f, g by n and express properties of the function $\sum(a, f)$ by appropriate Δ_0-formulas in which $U_{n+2}(X)$ occurs. Check that the Transfer Principle yields properties of the nonstandard extension ${}^*\sum$ that correspond to the items in the Proposition. □

Fix an uncountable cardinal number κ.

Recall that a collection of sets $\mathcal{F}$ is said to have the *finite intersection property* (FIP) if every finite subcollection of $\mathcal{F}$ has nonempty intersection.

4.13. **Definition.** The nonstandard extension ${}^*\colon U(X) \to U({}^*X)$ is said to be *κ-saturated* if, whenever I is a set of cardinality $< \kappa$ and $\{A_i \mid i \in I\}$ is a collection of internal sets with the finite intersection property, then $\bigcap\{A_i \mid i \in I\} \neq \emptyset$.

4.14. **Theorem** (Existence of κ-saturated nonstandard extensions)**.** *For every infinite set of individuals X there exists a κ-saturated nonstandard extension ${}^*\colon U(X) \to U({}^*X)$. Every such nonstandard extension is proper.*

Proof. Using basic model theory we may obtain a κ-saturated L-structure $\mathcal{M}$ and an elementary embedding F of $(U(X), \in, X)$ into $\mathcal{M}$. Apply the Mostowski collapsing construction as in the proof of Theorem 3.4 to obtain a nonstandard extension ${}^*\colon U(X) \to U({}^*X)$ with ${}^*X = I^{\mathcal{M}}$. We will show that this nonstandard extension is κ-saturated. Recall that this construction uses the substructure $\mathcal{M}_f$ of all strongly well-founded elements of $\mathcal{M}$ and an embedding G of $\mathcal{M}_f$ into $U({}^*X)$. The range of G is exactly the transitive substructure of $U({}^*X)$ consisting of the internal elements.

Suppose I is a set of cardinality less than κ and $\{A_i \mid i \in I\}$ is a collection of internal sets in $U({}^*X)$ that has the finite intersection property. For each $i \in I$, let $m_i \in \mathcal{M}_f$ be such that $G(m_i) = A_i$. We consider the $L(M)$-formulas xEm_i for $i \in I$. The fact that $\{A_i \mid i \in I\}$ has the FIP ensures that $\{xEm_i \mid i \in I\}$ is finitely satisfiable in $\mathcal{M}$. Since $\mathcal{M}$ is κ-saturated and $\operatorname{card}(I) < \kappa$, there exists $m \in \mathcal{M}$ such that $mE^{\mathcal{M}}m_i$ holds for all $i \in I$. It follows that m is strongly well founded, and we see that $G(m)$ is an element of the intersection of the collection $\{A_i \mid i \in I\}$. Therefore we have shown that the nonstandard extension ${}^*\colon U(X) \to U({}^*X)$ is κ-saturated.

Finally we show that any κ-saturated extension is proper. Let C be any countable, infinite subset of X and consider the internal sets $A_c = \{x \in {}^*C \mid x \neq c\}$ for $c \in C$. Since C is infinite, this collection of sets has the FIP. Since C is countable and the nonstandard extension is κ-saturated (with $\kappa > \omega$), we conclude that $\bigcap\{A_c \mid c \in C\}$ is nonempty. That is, there exists

$u \in {}^*C$ that is distinct from c for all $c \in C$, and hence the nonstandard extension is proper. □

4.15. **Proposition** (Comprehension principle).
(Assume κ-saturation.) Let A, B be internal sets in $U({}^*X)$ and let S be a subset of A with card$(S) < \kappa$*. For every function $\alpha\colon S \to B$, there exists an internal function $f\colon A \to B$ whose restriction to S is α.*

Proof. Assume A, B, S, α are as in the hypotheses. Let a_0 be a fixed element of A. For each $x \in S$ let

$$A_x = \{g \mid g\colon A \to B \text{ is internal and } g(x) = \alpha(x)\}.$$

Each A_x is an internal set. Moreover, the collection of sets $\{A_x \mid x \in S\}$ has the finite intersection property. Indeed, for each finite set $F \subseteq S$ the function $g\colon A \to B$ defined by

$$g(a) = \begin{cases} \alpha(a) & a \in F \\ \alpha(a_0) & \text{otherwise} \end{cases}$$

is internal and is an element of every A_x for $x \in F$. Using κ-saturation we conclude that $\bigcap_{x \in S} A_x$ contains some element g, which is necessarily an internal function $g\colon A \to B$ that extends α. □

The next result formulates the κ-saturation condition in a way that will be useful in many situations.

4.16. **Proposition.** *(Assume κ-saturation.) Let I, J be index sets, each of cardinality $< \kappa$. For each $i \in I$ and $j \in J$ let A_i and B_j be internal sets, and assume that all of the sets in $\{A_i \mid i \in I\} \cup \{B_j \mid j \in J\}$ have rank $\leq n$ in $U({}^*X)$. Then the following two conditions are equivalent:*
(1) $\bigcup\{A_i \mid i \in I\} \supseteq \bigcap\{B_j \mid j \in J\}$.
(2) There exist finite sets $I_0 \subseteq I$ and $J_0 \subseteq J$ such that
$\bigcup\{A_i \mid i \in I_0\} \supseteq \bigcap\{B_j \mid j \in J_0\}$.

Proof. Let the families $(A_i \mid i \in I)$ and $(B_j \mid j \in J)$ be as in the hypotheses. Note that every A_i and every B_j is a subset of ${}^*U_n(X)$. The equivalence between (1) and (2) follows immediately from an application of κ-saturation to the family of internal sets consisting of ${}^*U_n(X) \setminus A_i$ for $i \in I$ and B_j for $j \in J$. Note that (1) holds iff the intersection of these sets is empty. □

4.17. **Example.** (Assume ω_1-saturation.) Let A be a hyperfinite set in $U(^*X)$ and let $\mathcal{A}$ be the boolean algebra of all internal subsets of A. For each $B \in \mathcal{A}$ define

$$\mu(B) = \text{st}\,\Big(\frac{|B|}{|A|}\Big).$$

Note that this standard part is well defined because $0 \le |B| \le |A|$. Using properties of the standard part and of internal cardinalities of hyperfinite sets, it is easy to show that μ is a finitely additive probability measure on $\mathcal{A}$.

Moreover, because of ω_1-saturation, we have that μ satisfies the hypotheses of the Carathéodory Extension Theorem on $\mathcal{A}$. That is if $(A_n \mid n \in \mathbb{N})$ is a descending sequence in $\mathcal{A}$ and $A \in \mathcal{A}$ is equal to $\bigcap(A_n \mid n \in \mathbb{N})$, then $\mu(A_n)$ converges to $\mu(A)$ as $n \to \infty$. Indeed, this is true in a very strong way: using Proposition 4.16, there exists $k \in \mathbb{N}$ such that $A_n = A$ for all $n \ge k$ in $\mathbb{N}$.

Therefore we conclude that μ extends to a σ-additive probability measure defined on the σ-algebra of subsets of A generated by $\mathcal{A}$. This is an example of a *Loeb measure*, which play an important role in many applications of nonstandard analysis. These measures will be extensively studied later in the course.

4.18. **Exercises.** (Assume κ-saturation, $\kappa \ge \omega_1$.)
(1) Suppose S, T are subsets of $^*\mathbb{R}$ with $S < T$. (That is, for any $x \in S$ and $y \in T$, we have $x < y$.) Suppose further that S and T have cardinality $< \kappa$. Then there exists $a \in {}^*\mathbb{R}$ such that $S < a < T$. In particular, if each element of T is positive infinite, then there exists a positive infinite a with $a < T$. (Take $S = \mathbb{N}$).
(2) If A is an infinite internal set, then A has cardinality $\ge \kappa$.
(3) Suppose $(A_n \mid n \in \mathbb{N})$ is a sequence of distinct internal sets. If $A_0 \subset A_1 \subset \cdots \subset A_n \subset \ldots$, then $\bigcup\{A_n \mid n \in \mathbb{N}\}$ is external. Likewise, if $A_0 \supset A_1 \supset \cdots \supset A_n \supset \ldots$, then $\bigcap\{A_n \mid n \in \mathbb{N}\}$ is external.