

5. TOPOLOGY

Throughout this section we fix a set of individuals X that contains \mathbb{R} , and a proper nonstandard extension ${}^*U(X) \rightarrow U({}^*X)$.

In this section we show how nonstandard analysis can be used to characterize various topological notions, and we give some applications. The key concept is the *monad* of a point of T . As we will see, the system of monads determines the topology τ on T .

In order for this approach to work smoothly, we need to assume that our nonstandard extension is somewhat saturated, with the amount of saturation depending on the number of open sets.

5.1. Convention. Throughout this section, we work with a topological space (T, τ) that is in $U(X)$, and we assume that our nonstandard extension is at least $\text{card}(\tau)^+$ -saturated.

5.2. Definition. For $t \in T$ the τ -*monad* of t , denoted $\mu_\tau(t)$, is defined by

$$\mu_\tau(t) = \bigcap \left\{ {}^*O \mid O \text{ is } \tau\text{-open and } t \in O \right\}.$$

If no confusion will result we write $\mu(t)$ instead of $\mu_\tau(t)$.

For $t \in T$, we regard the elements of $\mu_\tau(t)$ as being *infinitesimally close* to t (more precisely, to *t) with respect to the topology τ .

5.3. Definition. An element $u \in {}^*T$ is τ -*nearstandard* if $u \in \mu_\tau(t)$ for some $t \in T$. We write $\text{ns}({}^*T)$ for the set of nearstandard elements of *T .

There are simple nonstandard characterizations of most separation axioms in topology. The next result does this for the Hausdorff condition.

5.4. Proposition. *The space (T, τ) is Hausdorff if and only if the intersection $\mu(t_1) \cap \mu(t_2)$ is empty for every distinct $t_1, t_2 \in T$.*

Proof. We show that for each $t_1, t_2 \in T$, the intersection $\mu(t_1) \cap \mu(t_2)$ is empty if and only if t_1 and t_2 can be separated by disjoint open sets. The Hausdorff condition is equivalent to the assertion that this is possible for all distinct t_1, t_2 .

For each $t \in T$ let \mathcal{N}_t be the collection of all open sets that have t as an element. Then $\mu(t)$ is the intersection of the family $\{{}^*O \mid O \in \mathcal{N}_t\}$, which is an intersection of fewer than $\text{card}(\tau)^+$ internal sets. (Indeed, they are

standard sets.) Note that the family is closed under finite intersections. It follows from $\text{card}(\tau)^+$ -saturation and this observation that $\mu(t_1) \cap \mu(t_2)$ is empty if and only if there exist $O_i \in \mathcal{N}_i$ for $i = 1, 2$ such that $*O_1 \cap *O_2$ is empty. This last condition is equivalent to $O_1 \cap O_2$ being empty. \square

5.5. Definition. For a Hausdorff topological space (T, τ) in $U(X)$, we define the *standard part map* $\text{st}: \text{ns}(*T) \rightarrow T$ by setting $\text{st}(u)$ equal to the unique $t \in T$ that satisfies $u \in \mu(t)$. (For non-Hausdorff spaces the standard part map would be multi-valued.)

5.6. Exercises. (1) Show that (T, τ) satisfies the T_0 separation axiom if and only if for every distinct $t_1, t_2 \in T$, either $*t_1 \notin \mu(t_2)$ or $*t_2 \notin \mu(t_1)$.

(2) Show that (T, τ) satisfies the T_1 separation property if and only if for every $t \in T$, the only standard element of $\mu(t)$ is $*t$.

The following lemma is very useful when applying nonstandard topological conditions in combination with the Transfer Principle.

5.7. Lemma (Infinitesimal open neighborhood). *For each $t \in T$ there exists a $*\text{open}$ set V (i.e., an element of $*\tau$) such that $*t \in V \subseteq \mu(t)$.*

Proof. Fix $t \in T$ and let \mathcal{N} be the collection of all open neighborhoods of t in (T, τ) . For each $O \in \mathcal{N}$ let $A(O) = \{V \in \tau \mid t \in V \subseteq O\}$. Note that the family of sets $\{A(O) \mid O \in \mathcal{N}\}$ has the finite intersection property, and hence that property also holds for the family $\{*A(O) \mid O \in \mathcal{N}\}$, all of whose members are internal sets. Hence, by our saturation assumption there exists V such that $V \in *A(O)$ for all $O \in \mathcal{N}$. That is, V is $*\text{open}$, has $*t$ as an element, and is a subset of $\mu(t) = \bigcap \{*O \mid O \in \mathcal{N}\}$. \square

5.8. Proposition. *Let $S \subseteq T$ and $t \in T$. Then*

(1) *t is in the interior of S if and only if $\mu(t) \subseteq *S$.*

(2) *t is in the closure of S if and only if $\mu(t) \cap *S \neq \emptyset$.*

Proof. We prove (1); (2) follows by applying (1) to the complement of S .

(\Rightarrow) If O is an open neighborhood of t that is contained in S , then

$$\mu(t) \subseteq *O \subseteq *S.$$

(\Leftarrow) (We give two proofs.)

By the previous lemma, there exists a $*\text{open}$ set V such that

$$*t \in V \subseteq \mu(t) \subseteq *S.$$

That is, we have $\exists V \in {}^*\tau$ (${}^*t \in V$ and $V \subseteq {}^*S$) and hence by the Transfer Principle we conclude $\exists O \in \tau$ ($t \in O$ and $O \subseteq S$), as desired.

Alternatively, if $\mu(t) \subseteq {}^*S$, then

$$\bigcap \{ {}^*O \mid O \text{ open, } t \in O \} \cap ({}^*T \setminus {}^*S) = \emptyset.$$

By Proposition 4.16 there exist finitely many open neighborhoods O_1, \dots, O_m of t such that

$${}^*O_1 \cap \dots \cap {}^*O_m \cap ({}^*T \setminus {}^*S) = \emptyset.$$

By transfer we have

$$O_1 \cap \dots \cap O_m \cap (T \setminus S) = \emptyset.$$

That is, $O_1 \cap \dots \cap O_m$ is an open neighborhood of t contained in S . \square

5.9. Remark. Proposition 5.8(2) states that if S is a subset of T , then $\text{st}({}^*S)$ is the closure of S . It follows from 5.8(1) that S is open if and only if $\mu(s) \subseteq {}^*S$ holds for all $s \in S$. Hence the monad system for (T, τ) determines τ . Likewise, if τ_1 and τ_2 are topologies on the same set T , then τ_1 is finer than τ_2 if and only if $\mu_{\tau_1}(t) \subseteq \mu_{\tau_2}(t)$ for all $t \in T$.

Next we give a nonstandard characterization of continuity. Suppose we have topological spaces (S, σ) and (T, τ) , and a function $f: S \rightarrow T$, all in the superstructure $U(X)$.

5.10. Proposition. *The function f is continuous at $x \in S$ if and only if ${}^*f(\mu_\sigma(x)) \subseteq \mu_\tau(f(x))$.*

Proof. First suppose f is continuous at x . For any τ -open set O with $f(x) \in O$, we have that x is in the σ -interior of $f^{-1}(O)$. Therefore $\mu_\sigma(x) \subseteq {}^*f^{-1}(O) = ({}^*f)^{-1}({}^*O)$. That is, *f maps $\mu_\sigma(x)$ into *O for every such O . Taking the intersection over all such O we conclude that *f maps $\mu_\sigma(x)$ into $\mu_\tau(f(x))$, as needed.

Conversely, suppose ${}^*f(\mu_\sigma(x)) \subseteq \mu_\tau(f(x))$. Using Lemma 5.7 we get a ${}^*\sigma$ -open set V such that ${}^*x \in V \subseteq \mu_\sigma(x)$. For any τ -open neighborhood O of $f(x)$ we have that *f maps V into *O . Applying the Transfer Principle to the existence of V we conclude that there exists a σ -open neighborhood V of x such that f maps V into O . In other words, x is in the interior of $f^{-1}(O)$. This implies that f is continuous at x . \square

5.11. Proposition. *If $A \subseteq {}^*T$ is internal, then $\text{st}(A)$ is closed.*

Proof. Take t in the closure of $\text{st}(A)$ in (T, τ) . So, for any open O such that $t \in O$, we have $O \cap \text{st}(A) \neq \emptyset$. Let $s \in O \cap \text{st}(A)$, so $\mu(s) \subseteq {}^*O$ and $\mu(s) \cap A \neq \emptyset$. Hence ${}^*O \cap A \neq \emptyset$ for each such O .

Our saturation assumption and the fact that $\{{}^*O \mid O \text{ is open and } t \in O\}$ is closed under finite intersections then yields:

$$\left(\bigcap \{{}^*O \mid O \text{ is open and } t \in O\}\right) \cap A \neq \emptyset.$$

That is, $\mu(t) \cap A \neq \emptyset$, and therefore $t \in \text{st}(A)$. □

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5.12. Proposition. *For every $S \subseteq T$, the following conditions are equivalent:*

- (1) S is compact.
- (2) For every $u \in {}^*S$ there exists $s \in S$ such that $u \in \mu(s)$.
- (3) S is closed and ${}^*S \subseteq \text{ns}({}^*T)$.

Proof. (1) \Rightarrow (2) If otherwise, S is compact yet there exists $u \in {}^*S$ with the property that $u \notin \mu(s)$ holds for every $s \in S$. Therefore, for each $s \in S$ we get an open set O_s such that $s \in O_s$ (and hence $\mu(s) \subseteq {}^*O_s$) but $u \notin {}^*O_s$. Then $\{O_s \mid s \in S\}$ is an open cover of S . Since S is compact, there exists a finite set $S_0 \subseteq S$ such that $S \subseteq \bigcup \{O_s \mid s \in S_0\}$. But then $u \in {}^*S \subseteq \bigcup \{{}^*O_s \mid s \in S_0\}$. Hence there exists $s \in S_0$ with $u \in {}^*O_s$, which is a contradiction.

(2) \Rightarrow (3) This is immediate using Proposition 5.8(2).

(3) \Rightarrow (1) Consider an open cover $\{O_i \mid i \in I\}$ of S . From (3) and 5.8(2) we conclude ${}^*S \subseteq \bigcup \{{}^*O_i \mid i \in I\}$. By Proposition 4.16, there must exist a finite set $I_0 \subseteq I$ with ${}^*S \subseteq \bigcup \{{}^*O_i \mid i \in I_0\} = {}^*(\{O_i \mid i \in I_0\})$. By transfer, $\{O_i \mid i \in I_0\}$ is a finite subcover of S . This shows that S is compact. □

If (T, τ) is a regular space, the preceding result can be strengthened. Recall that (T, τ) is *regular* if it is Hausdorff and for every $t \in T$ and every closed subset $C \subseteq T$, there exist disjoint open sets O and O' such that $t \in O$ and $C \subseteq O'$.

5.13. Proposition. *Assume (T, τ) is regular. If $A \subseteq \text{ns}(*T)$ is internal, then $\text{st}(A)$ is compact. In particular, if $S \subseteq T$ satisfies $*S \subseteq \text{ns}(*T)$ then the closure of S (which equals $\text{st}(*S)$) is compact.*

Proof. Let $\{O_i | i \in I\}$ be an open cover (in (T, τ)) of $\text{st}(A)$. Because (T, τ) is regular we can refine this open cover to $\{O'_j | j \in J\}$ which is an open cover of $\text{st}(A)$ and satisfies: for all $j \in J$ there exists $i \in I$ such that the closure of O'_j is contained in O_i . To achieve this, for each pair $(t, i) \in T \times I$ such that $t \in O_i$ choose an open set $O'_{(t,i)}$ that separates t from the complement of O_i . We can do this because (T, τ) is regular. The open cover we need is $\{O'_{(t,i)} | t \in O_i\}$; we reindex it as $\{O'_j | j \in J\}$ with J chosen so that the indexing avoids repetitions. Because $A \subseteq \text{ns}(*T)$ we have $A \subseteq \bigcup \{*O'_j | j \in J\}$. Since we are assuming $\text{card}(\tau)^+$ -saturation, this implies the existence of a finite set $J_0 \subseteq J$ such that $A \subseteq \bigcup \{*O'_j | j \in J_0\}$. From this we get

$$\text{st}(A) \subseteq \text{st}\left(\bigcup_{j \in J_0} *O'_j\right) \subseteq \bigcup_{j \in J_0} \text{st}(*O'_j).$$

Since $*O'_j$ is internal, $\text{st}(*O'_j)$ equals the closure of O'_j . So, for each $j \in J_0$ there is $i(j) \in I$ such that $O'_j \subseteq O_{i(j)}$. Thus

$$\text{st}(A) \subseteq \bigcup_{j \in J_0} \text{st}(*O'_j) \subseteq \bigcup_{j \in J_0} O_{i(j)}$$

That is, $\{O_{i(j)} | j \in J_0\}$ is a finite subcover of $\text{st}(A)$ from $\{O_i | i \in I\}$. Thus $\text{st}(A)$ is compact. \square

5.14. Example. We construct a non-regular, Hausdorff space (T, τ) in which the previous proposition is false.

Let $T = [0, 1]^2$ be the unit square. Let $L = [0, 1] \times \{0\}$. The basic τ -neighborhoods of a point (u, v) with $v > 0$ are the usual Euclidean neighborhoods; a basic neighborhood of $(u, 0)$ is of the form

$$O_r = \{(x, y) \in T | (x - u)^2 + y^2 < r \text{ and } y > 0\} \cup \{(u, 0)\}$$

where $r > 0$. Note that L is closed in T . Also, each $\{(u, 0)\}$ is relatively open in L , so the induced topology on L is the discrete topology (all sets are open). On $T \setminus L$, the topology is the usual Euclidean topology.

Evidently (T, τ) is Hausdorff and it is not compact. (For example, it has a closed subset L , which is not compact.) Note that $\text{ns}(*T)$ consists of the

points (u, v) where either $v = 0$ and u is standard, or $v > 0$. The standard part of a τ -nearstandard (u, v) is equal to $(\text{st}(u), \text{st}(v))$, where st is the standard part for the usual topology on $[0, 1]$.

A counterexample to the previous proposition is given by taking $S = T \setminus L$. Indeed, ${}^*S = \{(u, v) \in T \mid v > 0\}$, so each point of *S is τ -nearstandard. However, the closure of S is T , which is not compact.