## Spaces of continuous functions

For this subsection we fix a compact Hausdorff space $(T,\tau)$, and a metric space $(M,\rho)$, both in $U(X)$.

Let $C(T,M)$ denote the space of continuous functions from $T$ into $M$. We give $C(T,M)$ the topology of uniform convergence; this is the topology defined by the supremum metric, which is defined for $f,g\in C(T,M)$ by

$$d(f,g)=\sup\{\rho(f(t),g(t))|t\in T\}.$$

Note that because $(T,\tau)$ is compact and $f$ and $g$ are continuous, the set $\{\rho(f(t),g(t))|t\in T\}$ is bounded in $\mathbb{R}$.

5.16. **Notation.** If $a,b\in {}^*T$ we write $a\approx b$ if $\mathrm{st}(a)=\mathrm{st}(b)$. If $\alpha,\beta\in {}^*M$ we write $\alpha\approx\beta$ if ${}^*\rho(\alpha,\beta)\approx 0$. Note that both of these relations are equivalence relations.

5.17. **Definition.** An internal function $F\colon {}^*T\to {}^*M$ is said to be *$S$-continuous* if $a\approx b\Rightarrow F(a)\approx F(b)$ holds for all $a,b\in {}^*T$.

5.18. **Theorem.** *If $F\in {}^*C(T,M)$, then $F$ is nearstandard if and only if $F$ is $S$-continuous and $F({}^*t)\in \mathrm{ns}({}^*M)$ for all $t\in T$.*

*Proof.* ($\Rightarrow$) Assume $F$ is nearstandard and let $f\in C(T,M)$ be the standard part of $F$. This means that $F(a)\approx {}^*f(a)$ holds for all $a\in {}^*T$.

First we prove $F$ is $S$-continuous. Take any $a,b\in {}^*T$ such that $a\approx b$, and thus we have $t\in T$ that is the common $\tau$-standard part of $a$ and $b$. Since $f$ is continuous at $t$ we have $F(a)\approx {}^*f(a)\approx {}^*f(t)\approx {}^*f(b)\approx F(b)$.

Second, we note that $F({}^*t)$ is $\rho$-nearstandard in ${}^*M$ for every $t\in T$, since we have $F({}^*t)\approx {}^*f({}^*t)={}^*\big(f(t)\big)$.

($\Leftarrow$) Assume $F\colon {}^*T\to {}^*M$ is $S$-continuous and that $F({}^*t)\in \mathrm{ns}({}^*M)$ for all $t\in T$. Define $f\colon T\to M$ by $f(t)=\mathrm{st}(F({}^*t))$ for $t\in T$. We will show that $f$ is in $C(M,T)$ and that it is the standard part of $F$.

First we show that $f$ is continuous. Let $t\in T$ and $\epsilon>0$. Because $F$ is $S$-continuous, ${}^*\rho(F(a),F({}^*t))$ is infinitesimal for every $a\in\mu(t)$. Thus we have

$$\bigcap\{{}^*O\mid t\in O \text{ and } O \text{ is } \tau\text{-open}\}\subseteq\{a\in {}^*T\mid {}^*\rho(F(a),F({}^*t))<\epsilon\}.$$

Recall that we are assuming $\mathrm{card}(\tau)^+$-saturation. Therefore we may apply Proposition 4.16 to show that there exist $\tau$-open neighborhoods $O_1,\dots,O_n$

of $t$ such that ${}^*\rho(F(a), F({}^*t)) < \epsilon$ holds for every $a \in {}^*O_1 \cap \cdots \cap {}^*O_n$. Let $O$ be the open neighborhood $O_1 \cap \cdots \cap O_n$ of $t$ and consider any $s \in O$. We have

$$\begin{aligned} \rho(f(s), f(t)) &= {}^*\rho({}^*f(s), {}^*f(t)) \\ &\leq {}^*\rho({}^*f(s), F({}^*s)) + {}^*\rho(F({}^*s), F({}^*t)) + {}^*\rho(F({}^*t), {}^*f(t)) \\ &< \epsilon + {}^*\rho({}^*f(s), F({}^*s)) + {}^*\rho(F({}^*t), {}^*f(t)). \end{aligned}$$

From the definition of $f$ we conclude $\rho(f(s), f(t)) \leq \epsilon$ for every $s \in O$. Since $t$ and $\epsilon$ were arbitrary, this shows that $f$ is continuous on $T$.

Finally we must show that $f$ is the standard part of $F$. Take any $a \in {}^*T$ and let $t = \mathrm{st}(a)$. Then using the definition of $f$ and the fact that $F$ is $S$-continuous and that $f$ is continuous, we have $F(a) \approx F({}^*t) \approx {}^*f(t) \approx {}^*f(a)$, which means that ${}^*\rho(F(a), {}^*f(a))$ is infinitesimal. Since $d(F, {}^*f)$ is the internal supremum of ${}^*\rho(F(a), {}^*f(a))$ as $a$ ranges over ${}^*T$, we have that $d(F, {}^*f)$ is also infinitesimal, as needed. □

5.19. **Definition.** A subset $\mathcal{F}$ of $C(T, M)$ is *equicontinuous* if for every $t \in T$ and $\epsilon > 0$ there is an open neighborhood $O$ of $t$ such that $\rho(f(s), f(t) < \epsilon$ holds for all $s \in O$.

5.20. **Lemma.** *If $\mathcal{F} \subseteq C(T, M)$, then $\mathcal{F}$ is equicontinuous if and only if every $F \in {}^*\mathcal{F}$ is $S$-continuous.*

*Proof.* ($\Rightarrow$) Assume $\mathcal{F} \subseteq C(T, M)$ is equicontinuous and suppose $F \in {}^*C(T, M)$. Fix $a \approx b$ in ${}^*T$ and let $\epsilon > 0$ be standard. We may take $t \in T$ to be the common standard part of $a, b$. Let $O$ be an open neighborhood of $t$ that witnesses the equicontinuity of $\mathcal{F}$, in the sense that $\rho(f(s), f(t)) < \epsilon/2$ holds for every $f \in \mathcal{F}$ and every $s \in O$. Since $a, b \in \mu(t) \subseteq {}^*O$, we have by transfer that ${}^*\rho(F(a), F({}^*t)) < \epsilon/2$ and ${}^*\rho(F(b), F({}^*t)) < \epsilon/2$. The triangle inequality for ${}^*\rho$ yields ${}^*\rho(F(a), F(b)) < \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude $F(a) \approx F(b)$, as desired.

($\Leftarrow$) Assume every $F \in {}^*\mathcal{F}$ is $S$-continuous. Fix $t \in T$ and a standard $\epsilon > 0$. Consider the internal set

$$A = \{a \in {}^*T \mid {}^*\rho(F(a), F({}^*t)) < \epsilon \text{ for all } F \in {}^*\mathcal{F}\}.$$

Our assumption yields that $\mu(t) \subseteq A$. By Proposition 4.16, which applies since we are assuming $\mathrm{card}(\tau)^+$-saturation and $\mu(t)$ is equal to the intersection of at most $\mathrm{card}(\tau)$ many standard sets, there exist open sets $O_1, \ldots, O_n$

such that $t \in O_1 \cap \cdots \cap O_n$ and ${}^*O_1 \cap \cdots \cap {}^*O_n \subseteq A$. By transfer we have that $\rho(f(s), f(t)) < \epsilon$ holds for every $f \in \mathcal{F}$ and $s \in O_1 \cap \cdots \cap O_n$. Since $\epsilon > 0$ was arbitrary, this proves $\mathcal{F}$ is equicontinuous. □

Using the nonstandard analysis results above, we give a quick proof of the following standard theorem:

5.21. **Theorem** (Ascoli's Theorem)**.** *For $\mathcal{F} \subseteq C(T, M)$ the following are equivalent:*
*(1) $\mathcal{F}$ is relatively compact in $C(T, M)$.*
*(2) $\mathcal{F}$ is equicontinuous and $\{f(t)|f \in \mathcal{F}\}$ is relatively compact in $M$ for all $t \in T$.*

*Proof.* Since metric spaces are regular, Proposition 5.14 applies and shows that condition (1) is equivalent to
$(1')$ ${}^*\mathcal{F} \subseteq \text{ns}({}^*C(T, M)$.

For each $t \in T$ we let $\mathcal{F}_t = \{f(t)|f \in \mathcal{F}\}$. The Standard Definition Principle yields ${}^*\mathcal{F}_t = \{F({}^*t)|F \in {}^*\mathcal{F}\}$ for each $t \in T$.

By Theorem 5.18 we have that $(1')$ holds if and only if every $F \in {}^*\mathcal{F}$ is $S$-continuous and ${}^*\mathcal{F}_t \subseteq \text{ns}({}^*M)$ for all $t \in T$. The first part of this condition is equivalent to $\mathcal{F}$ being equicontinuous, by the previous Lemma. The second part is equivalent to $\mathcal{F}_t$ being relatively compact in $M$ for each $t \in T$, using Proposition 5.14 in $M$. Hence $(1')$ is equivalent to (2). □

We finish this subsection with another application of the nonstandard analysis characterization of compactness in spaces of continuous functions. In it we take $T$ to be $[0, 1]$ and $M$ to be $\mathbb{R}$, and we write $C[0, 1]$ for $C([0, 1], \mathbb{R})$.

Fix a continuous function $K\colon [0, 1]^2 \to \mathbb{R}$. Consider the linear operator $\Phi\colon C[0, 1] \to C[0, 1]$ defined by

$$\Phi(f)(t) = \int_0^1 K(t, s)f(s)ds.$$

5.22. **Theorem.** *The operator $\Phi$ on $C(T, M)$ is compact; that is, the set*

$$R = \{\Phi(f)|f \in C[0, 1] \text{ and } ||f(t)|| \leq 1\}$$

*is relatively compact in $C[0, 1]$.*

*Proof.* Since $K$ is continuous on $[0,1]^2$, which is compact, there exists $C>0$ such that $|K(s,t)| \leq C$ for each $s,t \in [0,1]$. Also, $K$ is uniformly continuous on $[0,1]^2$, which yields that ${}^*K(a,b) \approx {}^*K(a',b')$ in ${}^*R$ whenever $a \approx a'$ and $b \approx b'$ in ${}^*[0,1]$.

Using Proposition 5.14 in the regular space $C[0,1]$, it suffices to show ${}^*R \subseteq \mathrm{ns}({}^*C[0,1])$. By Theorem 5.18 it therefore suffices to show that for each $t \in T$ the set $\{G(t) \mid G \in {}^*R\}$ is contained in $\mathrm{ns}({}^*\mathbb{R}) = \mathrm{fin}({}^*\mathbb{R})$ and that each $G \in {}^*R$ is $S$-continuous.

Let $G \in {}^*R$. There is a ${}^*$continuous $F\colon {}^*[0,1] \to {}^*\mathbb{R}$ satisfying $|F(u)| \leq 1$ for all $u \in {}^*[0,1]$, such that $G = {}^*\Phi(F)$.

First we show that $G(t) \in {}^*[-C,C] \subseteq \mathrm{fin}({}^*\mathbb{R})$ for all $t \in [0,1]$. For $t \in [0,1]$ we have

$$|G(t)| = \left| \int_0^1 {}^*K(t,v)F(v)dv \right| \leq \int_0^1 |{}^*K(t,v)||F(v)|dv \leq \int_0^1 Cdv = C.$$

It remains to show that $G$ is S-continuous. Let $a \approx b$ in ${}^*[0,1]$. Since $K$ is uniformly continuous,

$$\{{}^*K(a,v) - {}^*K(b,v) : v \in {}^*[0,1]\} \subseteq \mu(0).$$

By underspill there exists a positive infinitesimal $\delta$, such that $|{}^*K(a,v) - {}^*K(b,v)| \leq \delta$ for all $v \in {}^*[0,1]$. Then we have

$$\begin{aligned}
|G(a) - G(b)| &= \left| \int_0^1 {}^*K(a,v)F(v)dv - \int_0^1 {}^*K(b,v)F(v)dv \right| \\
&= \left| \int_0^1 \left[{}^*K(a,v) - {}^*K(b,v)\right] F(v)dv \right| \\
&\leq \int_0^1 |{}^*K(a,v) - {}^*K(b,v)||F(v)|dv \\
&\leq \int_0^1 \delta dv = \delta \approx 0.
\end{aligned}$$

This finishes the proof. $\square$