Quotient spaces

If (S, σ) and (T, τ) are topological spaces and $\pi \colon S \to T$ is a surjective function, then π is called a *quotient map* if for every $O \subseteq T$ we have that O is τ-open if and only if $\pi^{-1}(O)$ is σ-open.

Note that if (S, σ) is given and $\pi \colon S \to T$ is surjective, then there is a unique topology τ on T such that π is a quotient map from (S, σ) onto (T, τ). Namely, we declare $O \subseteq T$ to be τ-open if and only if $\pi^{-1}(O)$ is σ-open. (It is easy to check that this defines a topology on T.) This is a common way to define new topological spaces.

In the rest of this subsection we assume that (S, σ) and (T, τ) are topological spaces in $U(X)$ and that $\pi \colon S \to T$ is a quotient map. Of course we also assume that our saturation assumption applies to both spaces. That is, we assume our nonstandard extension is κ-saturated for some κ such that $\kappa > \mathrm{card}(\sigma)$ and $\kappa > \mathrm{card}(\tau)$.

Our most important task is to determine the monad system of the quotient space (T, τ) in terms of the monad system of (S, σ) and the function ${}^*\pi \colon {}^*S \to {}^*T$. Since π is continuous, for each $t \in T$ we have

$$\mu_\tau(t) \supseteq \bigcup \{ {}^*\pi(\mu_\sigma(s)) \mid s \in S \text{ and } \pi(s) = t \}.$$

Indeed, we know that π is continuous if and only if this containment holds for all $t \in T$. It is convenient to introduce the notation

$$M_\pi(t) = \bigcup \{ {}^*\pi(\mu_\sigma(s)) \mid s \in S \text{ and } \pi(s) = t \}$$

for all $t \in T$.

Although $(M_\pi(t) \mid t \in T)$ need not equal $(\mu_\tau(t) \mid t \in T)$, the τ-open sets are nonetheless determined by the sets $M_\pi(t)$ in the same way as they are determined by the monads $\mu_\tau(t)$, as we show next.

5.23. **Proposition.** *If $W \subseteq T$, then W is τ-open if and only if $M_\pi(t) \subseteq {}^*W$ for all $t \in W$.*

Proof. ($\Rightarrow$) If W is τ-open and $t \in W$, then we know ${}^*W \supseteq \mu_\tau(t) \supseteq M_\pi(t)$.

($\Leftarrow$) Suppose $W \subseteq T$ and $M_\pi(t) \subseteq {}^*W$ for all $t \in W$. Suppose $s \in \pi^{-1}(W)$. Then $t = \pi(s) \in W$, so ${}^*\pi(\mu_\sigma(s)) \subseteq M_\pi(t) \subseteq {}^*W$. This implies ${}^*(\pi^{-1}(W)) = ({}^*\pi)^{-1}({}^*W) \supseteq \mu_\sigma(s)$ for every $s \in \pi^{-1}(W)$. Hence $\pi^{-1}(W)$ is σ-open, which implies that W is τ-open since π is a quotient map. $\square$

5.24. **Definition.** We say the quotient map π satisfies the *monad condition* if $\mu_\tau(t) = M_\pi(t)$ holds for all $t \in T$.

5.25. **Proposition.** *If π satisfies the monad condition and (S, σ) is Hausdorff, then (T, τ) is also Hausdorff.*

Proof. Fix distinct $t_1, t_2 \in T$. If $s_1, s_2 \in S$ satisfy $\pi(s_1) = t_1$ and $\pi(s_2) = t_2$, then $s_1 \neq s_2$ and hence $\mu_\sigma(s_1) \cap \mu_\sigma(s_2) = \emptyset$. Letting s_1, s_2 vary, we conclude $M_\pi(t_1) \cap M_\pi(t_2) = \emptyset$. Using the monad condition we have $\mu_\tau(t_1) \cap \mu_\tau(t_2) = \emptyset$. Since this holds for all distinct $t_1, t_2 \in T$, it follows that (T, τ) is Hausdorff. $\square$

5.26. **Proposition.** *The quotient map π satisfies the monad condition if and only if ${}^*\pi$ maps $\mathrm{ns}({}^*S)$ onto $\mathrm{ns}({}^*T)$.*

Proof. Note that since π is continuous, ${}^*\pi$ always maps $\mathrm{ns}({}^*S)$ *into* $\mathrm{ns}({}^*T)$.
($\Rightarrow$) Say $v \in \mathrm{ns}({}^*T)$ with $t = \mathrm{st}(v)$. The monad condition implies $v \in {}^*\pi(\mu_\sigma(s))$ for some $s \in S$ for which $\pi(s) = t$. In particular, $v \in {}^*\pi(\mathrm{ns}({}^*S))$.
($\Leftarrow$) Let $t \in T$ and $v \in \mu_\tau(t)$. We must show $v \in M_\pi(t)$. Our assumption yields $u \in \mathrm{ns}({}^*S)$ such that ${}^*\pi(u) = v$. If $s = \mathrm{st}(u)$, then continuity of π implies $t = \pi(s)$ and thus we have $v \in {}^*\pi(\mu_\sigma(s)) \subseteq M_\pi(t)$. $\square$

5.27. **Corollary.** *If S is covered by sets of the form $\pi^{-1}(W)$ that are open and relatively compact, then the quotient map π satisfies the monad condition.*

Proof. Suppose $v \in \mathrm{ns}({}^*T)$ and $t \in T$ is the standard part of v. There exists $s \in S$ such that $\pi(s) = t$. By assumption there exists $W \subseteq T$ such that $s \in \pi^{-1}(W)$ and such that $\pi^{-1}(W)$ is open and relatively compact. It follows that W is open, and hence $\mu_\tau(t) \subseteq {}^*W$. Therefore

$$({}^*\pi)^{-1}(v) \subseteq ({}^*\pi)^{-1}(\mu_\tau(t)) \subseteq ({}^*\pi)^{-1}({}^*W) = {}^*(\pi^{-1}(W)) \subseteq \mathrm{ns}({}^*S)$$

from which it follows that $v = {}^*\pi(u)$ for some $u \in \mathrm{ns}({}^*S)$. The previous result implies that π satisfies the monad condition. $\square$

5.28. **Remark.** The condition on π that is assumed in the previous result is equivalent to the statement that (S, σ) and (T, τ) are locally compact and π is a proper map (*i.e*, $\pi^{-1}(K)$ is compact in S whenever K is compact in T).

We have need for the following topological space when we discuss continuity of the roots of a complex polynomial as a function of the coefficients.

5.29. **Example.** Let T be any set. A *multiset of cardinality n from T* is a pair (F, α) where $F \subseteq T$ is finite and α is a function $\alpha\colon F \to \{1, 2, \dots\}$ such that $\sum_{z \in F} \alpha(z) = n$.

Let $\mathbb{C}$ be the complex plane with the usual Euclidean topology, and fix $n \geq 1$. Let $\mathcal{M}_n$ denote the collection of all multisets of cardinality n from $\mathbb{C}$. Let $\pi_n\colon \mathbb{C}^n \to \mathcal{M}_n$ be defined by setting $\pi_n(z_1, \dots, z_n) = (F, \alpha)$ where $F = \{z_1, \dots, z_n\}$ and for each $z \in F$, $\alpha(z)$ is the number of i for which $z = z_i$. We give $\mathcal{M}_n$ the quotient topology induced by the usual topology on $\mathbb{C}^n$ and the surjective map π_n.

For each $r > 0$ let $W_r = \{(F, \alpha) \in \mathcal{M}_n \mid |z| < r \text{ for all } z \in F\}$. Then $\pi_n^{-1}(W_r) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < r \text{ for all } j = 1, \dots, n\}$. For each $r > 0$, this set is open and relatively compact in $\mathbb{C}^n$, and $\mathbb{C}^n$ is covered by this family of sets. By Corollary 5.27, the quotient map π_n satisfies the monad condition. Further, we have that both $\mathbb{C}^n$ and $\mathcal{M}_n$ are locally compact and that π_n is a proper map.

5.30. **Exercise.** Note that ${}^*\mathcal{M}_n$ is the collection of multisets of cardinality n from ${}^*\mathbb{C}$. Note also that $(F, \alpha) \in {}^*\mathcal{M}_n$ is nearstandard if and only if every element of F is nearstandard in ${}^*\mathbb{C}$. Let

$$A = \{(F, \alpha) \in {}^*\mathcal{M}_n \mid \alpha(z) = 1 \text{ for all } z \in F\}.$$

For nearstandard $(F, \alpha) \in A$, show that the standard part of (F, α) is the multiset (G, β) of cardinality n from $\mathbb{C}$, where $G = \{\operatorname{st}(z) | z \in F\}$ and $\beta(\operatorname{st}(z)) = |\{w \in F \mid w \approx z\}|$ for all $z \in F$. Show that A intersects every monad in $\mathcal{M}_n$. (See the discussion of such internal sets in the next subsection.)

Internal dense sets

5.31. **Terminology.** An *S-dense* set for a topological space (T, τ) is an internal subset of ${}^*T$ such that $A \cap \mu(t) \neq \emptyset$ holds for every $t \in T$.

5.32. **Proposition.** *Let (T, τ) be regular and let A be any S-dense internal subset of ${}^*T$. Then for any closed $C \subseteq T$, there is an internal $B \subseteq A$ with $\operatorname{st}(B) = C$.*

Proof. Let C be any closed subset of T. Let $B_1 \subseteq A$ be any set that selects one point from each monad $\mu(t)$, $t \in C$, so $\mathrm{st}(B_1) = C$. (B_1 is not necessarily internal.)

Next, let $\mathcal{F}$ be the collection of open O such that $C \cap \overline{O} = \emptyset$. Note $T \backslash C = \cup \mathcal{F}$ by the regularity of (T, τ). We want to use saturation to get an internal $B \subseteq A$ such that $B_1 \subseteq B$ and such that $O \in \mathcal{F}$ implies ${}^*O \cap B = \emptyset$. These conditions on B are finitely satisfiable. Indeed, given $u_1, \ldots, u_m \in B_1$ and $O_1, \ldots, O_n \in \mathcal{F}$, take

$$B = \left({}^*T \setminus (\cup_{i=1}^{n} {}^*O_i)\right) \cup \{u_1, \ldots, u_n\}.$$

Then B contains $u_1, \ldots, u_m$ and is disjoint from ${}^*O_1, \ldots, {}^*O_m$ as long as $u_i \notin {}^*O_j$ for each i, j. But $u_i \in \mu(\mathrm{st}(u_i)) \subseteq {}^*T \setminus ({}^*\overline{O_j}) \subseteq {}^*T \setminus {}^*O_j$.

Evidently $\mathrm{st}(B) \supseteq \mathrm{st}(B_1) = C$. So it remains only to show $\mathrm{st}(B) \subseteq C$. If $t \in T \setminus C$, regularity of (T, τ) implies there is an open neighborhood O of t whose closure is disjoint from C. Our choice of B ensures that B is disjoint from ${}^*O \supseteq \mu(t)$ and hence B is disjoint from $\mu(t)$. That is, $t \notin \mathrm{st}(B)$. □

5.33. **Remark.** The previous results show that if (T, τ) is a Hausdorff, regular topological space and A is an S-dense internal subset of ${}^*T$, then the topology τ on T is completely determined by the standard part map restricted to A (more precisely, to $A \cap \mathrm{ns}({}^*T)$). Namely, the family of τ-closed subsets of T is equal to the family of sets of the form $\mathrm{st}(B)$ where B ranges over the internal subsets of A.