

6. DERIVATIVES AND DIFFERENTIAL EQUATIONS

Consider an open interval (a, b) in \mathbb{R} and a function $f: (a, b) \rightarrow \mathbb{R}$. Using the ideas in section 1 we can easily give a nonstandard characterization of the derivative (since it depends directly on limits), as follows:

6.1. **Exercise.** Let $a < c < b$; the function f is differentiable at c with derivative D if and only if for all nonstandard $u \in \mu(c)$ we have

$$\frac{{}^*f(u) - {}^*f(c)}{u - c} \approx D.$$

A little work translates this into the following more symmetric condition:

6.2. **Exercise.** Let $a < c < b$; the function f is differentiable at c if and only if for all nonstandard u, u' in $\mu(c)$ we have

$$\frac{{}^*f(u) - {}^*f(c)}{u - c} \approx \frac{{}^*f(u') - {}^*f(c)}{u' - c}.$$

(In other words, f is differentiable at c iff the difference quotient with one end point at c is nearly constant on the monad of c .)

Note that in order to make use of 6.1 to solve 6.2, you have to show that the difference quotients in 6.2 are finite. This can be done using an overspill argument. The assumption in 6.2 implies that for every infinitesimal $\delta > 0$, if $c - \delta < u, u' < c + \delta$ and u, u' do not equal c , then

$$\left| \frac{{}^*f(u) - {}^*f(c)}{u - c} - \frac{{}^*f(u') - {}^*f(c)}{u' - c} \right| \leq 1.$$

By overspill, this internal condition holds for some standard $\delta > 0$. Taking $c - \delta < u' < c + \delta$ to be standard and u nonstandard in $\mu(c)$, we see that the quotients

$$\frac{{}^*f(u) - {}^*f(c)}{u - c}$$

must all be finite.

The following result gives a similar looking characterization of *continuous* differentiability. The differences between the nonstandard characterization in this result and in the one in the previous exercise are worth thinking about.

6.3. Proposition. *Let f be any function from (a, b) to \mathbb{R} . The following conditions are equivalent:*

- (1) *The derivative f' exists and is continuous on (a, b) .*
- (2) *For any u, u', v, v' in ${}^*(a, b)$ with $u \approx u' \approx v \approx v'$, $u \neq v$, $u' \neq v'$ and $a < \text{st}(u) < b$, we have*

$$\frac{{}^*f(u) - {}^*f(v)}{u - v} \approx \frac{{}^*f(u') - {}^*f(v')}{u' - v'}.$$

Proof. (1) \Rightarrow (2) Assume f' exists and is continuous on (a, b) . Take c in (a, b) and let $u \neq v$ and $u' \neq v'$ all be elements of $\mu(c)$. Use the Mean Value Theorem internally to get w between u, v and w' between u', v' such that

$${}^*f'(w) = \frac{{}^*f(u) - {}^*f(v)}{u - v} \text{ and } {}^*f'(w') = \frac{{}^*f(u') - {}^*f(v')}{u' - v'}.$$

Now w and w' both come from $\mu(c)$, so the continuity of f' ensures that ${}^*f'(w) \approx f'(c) \approx {}^*f'(w')$ and hence we have (2).

(2) \Rightarrow (1) Assume that (2) holds and consider any c in (a, b) . Taking $v = v' = c$ and using 6.2, we see that f is differentiable at c . That is, f' exists on all of (a, b) .

It remains to show that f' is continuous. Let $c \in (a, b)$ and take nonstandard $u \in \mu(c)$. Let ϵ be a positive infinitesimal. Applying the Transfer Principle to the definitions of ${}^*f'(u)$ and ${}^*f'(c)$ we obtain $\delta > 0$ in ${}^*\mathbb{R}$ such that

$$\forall v \in {}^*(a, b) \left[0 < |u - v| < \delta \rightarrow \left| \frac{{}^*f(u) - {}^*f(v)}{u - v} - {}^*f'(u) \right| < \epsilon \right].$$

and

$$\forall u' \in {}^*(a, b) \left[0 < |u' - c| < \delta \rightarrow \left| \frac{{}^*f(u') - {}^*f(c)}{u' - c} - f'(c) \right| < \epsilon \right]$$

Choose v, u' satisfying $0 < |u - v| < \delta$ and $0 < |u' - c| < \delta$. Then the triangle inequality yields

$$\begin{aligned} |{}^*f'(u) - {}^*f'(c)| &\leq \left| {}^*f'(u) - \frac{{}^*f(u) - {}^*f(v)}{u - v} \right| + \left| \frac{{}^*f(u) - {}^*f(v)}{u - v} - \frac{{}^*f(u') - {}^*f(c)}{u' - c} \right| \\ &\quad + \left| \frac{{}^*f(u') - {}^*f(c)}{u' - c} - {}^*f'(c) \right| \end{aligned}$$

This shows that ${}^*f'(u) \approx {}^*f'(c)$, whenever $u \in \mu(c)$. Therefore f' is continuous. \square

6.4. Exercise. Show that condition (2) in the previous proposition is equivalent to each of the following standard conditions (which thus also give characterizations of continuous differentiability).

3. f is *locally uniformly differentiable* on (a, b) ; that is for all $c \in (a, b)$

$$\lim_{\substack{x, y \rightarrow c \\ x \neq y}} \frac{f(x) - f(y)}{x - y} = f'(c).$$

4. f is *uniformly differentiable on closed subintervals* of (a, b) , that is for any $d, e \in (a, b)$ with $d < e$, and for all $\epsilon > 0$, there exists $\delta > 0$ such that for any $c \in [d, e]$ and for any z with $0 < |z| < \delta$ we have

$$\left| \frac{f(c+z) - f(c)}{z} - f'(c) \right| < \epsilon$$

It is possible to characterize C^n functions ($n \geq 2$) in a similar manner. Let f be a function on (a, b) , and let $u = (u_0, \dots, u_n)$ be a sequence of $n + 1$ distinct elements of (a, b) . There is a unique polynomial p_u of degree n satisfying $p_u(u_i) = f(u_i)$ for all $i = 0, \dots, n$. There exist unique coefficients a_0, \dots, a_n so that this polynomial can be written in the form

$$p_u(x) = a_0 + a_1(x - u_0) + a_2(x - u_0)(x - u_1) + \cdots + a_n \prod_{i=0}^{n-1} (x - u_i).$$

Evidently a_k can be obtained using induction on k since it solves the equation

$$f(u_k) = a_0 + a_1(u_k - u_0) + a_2(u_k - u_0)(u_k - u_1) + \cdots + a_k \prod_{i=0}^{k-1} (u_k - u_i).$$

This also makes it clear that a_k only depends on u_0, \dots, u_k for all $k = 0, \dots, n$. We denote a_k by $\delta^k f[u_0, \dots, u_k]$.

Note that

$$\delta^1 f[u_0, u_1] = \frac{f(u_1) - f(u_0)}{u_1 - u_0}$$

which is the difference quotient used for treating the first derivative of f . For $n > 1$ it is easy to show that $\delta^n f[u_0, \dots, u_n]$ is a higher order difference quotient of f .

6.5. Theorem. A function $f: (a, b) \rightarrow \mathbb{R}$ is C^n on (a, b) if and only if for all $c \in (a, b)$ and for all $u, v \in \mu(c)^{n+1}$, with u (respectively v) having distinct coordinates, one has

$$\delta^n f[u_0, \dots, u_n] \approx \delta^n f[v_0, \dots, v_n].$$

In that case, these expressions are finite and their common standard part is equal to $f^{(n)}(c)/n!$

Proof. See Chapter 8 of Keith Stroyan's online book *Foundations of Infinitesimal Calculus*¹ □

Now we will prove a classical existence result for continuous first order differential equations.

6.6. Theorem (Peano Existence Theorem). *Let $g: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded, continuous function. Then for any $a \in \mathbb{R}$, there is a differentiable function $f: [0, 1] \rightarrow \mathbb{R}$ such that $f(0) = a$ and $f'(t) = g(t, f(t))$ for all $t \in [0, 1]$*

Proof. Fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $\Delta t = 1/N$ and $T = \{0, \Delta t, 2\Delta t, \dots, N\Delta t\}$. Note that T is internal and S -dense in $[0, 1]$. By the Transfer Principle there is an internal function $Y: T \rightarrow {}^*\mathbb{R}$ such that $Y(0) = a$ and

$$Y((k+1)\Delta t) = Y(k\Delta t) + {}^*g(k\Delta t, Y(k\Delta t))\Delta t, \text{ for } 0 \leq k < N \text{ in } {}^*\mathbb{N}.$$

Step 1. First we prove that Y is S -continuous and $Y(t)$ is finite for all $t \in T$. Fix a bound M for g . Let $0 \leq k < l \leq N$. Then, by transfer, we have

$$\begin{aligned} |Y(l\Delta t) - Y(k\Delta t)| &= \left| \sum_{n=k}^{l-1} {}^*g(n\Delta t, Y(n\Delta t))\Delta t \right| \\ &\leq \sum_{n=k}^{l-1} |{}^*g(n\Delta t, Y(n\Delta t))\Delta t| \\ &\leq M \sum_{n=k}^{l-1} \Delta t = M(l-k)\Delta t \approx 0. \end{aligned}$$

This proves that Y is S -continuous. By taking $k = 0$ we also see that $|Y(l\Delta t)| \leq a + Ml\Delta t \leq a + M$. Hence $Y(l\Delta t)$ is finite.

Now we can define $f: [0, 1] \rightarrow \mathbb{R}$ by $f(r) = \text{st}(Y(t))$ where t is any element in $T \cap \mu(r)$; for definiteness, take $t = \min\{s \in T \mid r \leq s\}$. Clearly $f(0) = a$. It follows from the inequalities proved above for Y that f is continuous.

¹available from Stroyan's website at <http://www.math.uiowa.edu/~stroyan> by following the path `/InfsmlCalculus/FoundationsT0C.htm`. Also see his calculus lectures using infinitesimals at `/InfsmlCalculus/InfsmlCalc.htm`.

It remains to show that f is differentiable on $[0, 1]$ and that $f'(t) = g(t, f(t))$ for all $t \in [0, 1]$. Using the Fundamental Theorem of Calculus, this is equivalent to the integral equation

$$f(t) = a + \int_0^t g(s, f(s)) ds \quad \text{for all } t \in [0, 1].$$

Step 2. We prove this integral equation. Let $H: T \rightarrow {}^*\mathbb{R}$ be defined by $H(k\Delta t) = {}^*g(k\Delta t, Y(k\Delta t))$ and let the continuous function $h: [0, 1] \rightarrow \mathbb{R}$ be defined by $h(r) = g(r, f(r))$. Then we have

$$\begin{aligned} H(k\Delta t) &= {}^*g(k\Delta t, Y(k\Delta t)) \approx g(\text{st}(k\Delta t), f(\text{st}(k\Delta t))) \\ &= h(\text{st}(k\Delta t)) \approx {}^*h(k\Delta t). \end{aligned}$$

Hence $\{|H(n\Delta t) - {}^*h(n\Delta t)| \mid n = 0, \dots, N\}$ is a hyperfinite set of infinitesimals. Therefore this set has a maximum element δ which is infinitesimal.

This implies that for each $m \in {}^*\mathbb{N}$ with $0 \leq m \leq N$ we have

$$\begin{aligned} \left| \sum_{n=0}^m H(n\Delta t)\Delta t - \sum_{n=0}^m {}^*h(n\Delta t)\Delta t \right| &\leq \sum_{n=0}^m |H(n\Delta t) - {}^*h(n\Delta t)|\Delta t \\ &\leq \sum_{n=0}^m \delta\Delta t = (m+1)\delta\Delta t \approx 0. \end{aligned}$$

Therefore

$$\begin{aligned} f(t) &\approx Y(k\Delta t) = a + \sum_{n=0}^{k-1} {}^*g(n\Delta t, Y(n\Delta t))\Delta t \\ &\approx a + \sum_{n=0}^{k-1} H(n\Delta t)\Delta t \approx a + \sum_{n=0}^{k-1} {}^*h(n\Delta t)\Delta t \\ &\approx a + \int_0^t h(s)ds. \end{aligned}$$

The last step follows from the definition of the integral of the continuous function h and the nonstandard characterization of limits. Since the numbers at the beginning and end of this chain are standard, we conclude

$$f(t) = a + \int_0^t h(s)ds$$

which completes the proof. \square