7. Polynomials and Linear Algebra

In this section we apply nonstandard analysis to the study of roots of polynomials and to the structure of linear operators on finite dimensional vector spaces.

Roots of complex polynomials

Fix $n \in \mathbb{N}$ and let $\mathcal{P}_n \subseteq \mathbb{C}[z]$ be the set of polynomials over $\mathbb{C}$ of degree n. Recall that $\mathcal{M}_n$ is the space of multisets of cardinality n from $\mathbb{C}$. (See Example 5.29.) We consider the map $\mathcal{R}_n \colon \mathcal{P}_n \to \mathcal{M}_n$ which takes each polynomial to its set of roots, with multiplicities counted. Our objective in this subsection is to use nonstandard analysis to prove the continuity of this map. Indeed, we do something more general.

Each polynomial can be identified with its sequence of coefficients. This identifies $\mathcal{P}_n$ with a subset of $\mathbb{C}^{n+1}$ and we give it the subspace topology. Thus two polynomials are considered to be close if their coefficients are near each other in $\mathbb{C}$, degree by degree.

The nonstandard extension ${}^*\mathcal{P}_n$ is the set of polynomials over ${}^*\mathbb{C}$ of degree n. Moreover, ${}^*\mathcal{M}_n$ is the space of multisets of cardinality n over ${}^*\mathbb{C}$, and the map ${}^*\mathcal{R}_n$ takes each (nonstandard) polynomial to its multiset of roots in the algebraically closed field ${}^*\mathbb{C}$.

In order to prove the continuity of $\mathcal{R}_n$ we need to take polynomials $p \in \mathcal{P}_n$ and $q \in {}^*\mathcal{P}_n$, with the representations

$$p(z) = \sum_{i=0}^{n} a_i z^i \text{ and } q(z) = \sum_{i=0}^{n} b_i z^i$$

and satisfying $b_i \approx a_i$ for $i = 0, \ldots, n$, and we must show ${}^*\mathcal{R}_n(q) \approx \mathcal{R}(p)$ holds in ${}^*\mathcal{M}_n$.

The elementary algebraic argument we use applies more generally. We consider internal polynomials $p, q \in {}^*\mathcal{P}_n$ represented as above with coefficients in ${}^*\mathbb{C}$, such that $a_i \approx b_i$ for $i = 0, \ldots, n$. We also assume that p, q have at least one coefficient that is not infinitesimal. Our analysis relates the roots of p to the roots of q.

Obviously the problem is unchanged if we rescale p and q so that all coefficients are finite and for some $j = 1, \ldots, n$ we have $a_j \approx 1 \approx b_j$. (Divide by a coefficient with largest absolute value, which will not be infinitesimal.

Therefore the coefficients remain infinitesimally close, degree by degree.) Assume that this has been done, and define $p_0(z) \in \mathbb{C}[z]$ by

$$p_0(z) = \sum_{i=0}^{n} \mathrm{st}(a_i) z^i = \sum_{i=0}^{n} \mathrm{st}(b_i) z^i.$$

The next result relates the roots of p (and q) in ${}^*\mathbb{C}$ to the roots of p_0 in $\mathbb{C}$. Note that if the leading coefficient(s) of p, q are infinitesimal, then p_0 has degree $< n$, so it is not in $\mathcal{P}_n$. The next result shows that the amount by which the degree drops below n is exactly the number of infinite roots of p, q.

7.1. **Theorem.** *Suppose* $p(z) \in {}^*\mathcal{P}_n$*, with* $p(z) = a_n z^n + \cdots + a_1 z + a_0$*; assume the coefficients of* p *are finite,* $a_n \neq 0$*, and some* a_j *is not infinitesimal. List the roots of* p *in* ${}^*\mathbb{C}$ *(with repetitions) as* $\alpha_1, \ldots, \alpha_q, \beta_{q+1}, \ldots, \beta_n$*, where* $0 \leq q \leq n$*, each* α_i *is finite, and each* β_j *is infinite. Set* $r_j = \mathrm{st}(a_j)$ *for* $j = 0, 1, \ldots, n$ *and* $u_i = \mathrm{st}(\alpha_i)$ *for* $i = 1, \ldots, q$*. Then the polynomial* $p_0(z) = r_n z^n + \cdots + r_1 z + r_0$ *in* $\mathbb{C}[z]$ *has degree* q *and its roots are* $u_1, \ldots, u_q$*.*

Proof. Letting $k = a_n$, we have

$$p(z) = (z - \alpha_1) \ldots (z - \alpha_q)[k(z - \beta_{q+1}) \ldots (z - \beta_n)].$$

Let $S \subseteq {}^*\mathbb{C}$ be the set of all finite $z \in {}^*\mathbb{C}$ that are outside all of the monads $\mu(u_j)$, $j = 1, \ldots, q$. For all $z \in S$, we have $p(z) \approx p_0(z)$, since

$$p(z) - p_0(z) = \sum_{i=0}^{n} (a_i - \mathrm{st}(a_i)) z^i \approx 0.$$

By choice, $z - \alpha_i$ is not infinitesimal when $z \in S$, for $i = 1, \ldots, q$. Therefore, for such z we have

$$\frac{p_0(z)}{(z - u_1) \ldots (z - u_q)} \approx \frac{p(z)}{(z - \alpha_1) \ldots (z - \alpha_q)} = T(z),$$

where $T(z) = k(z - \beta_{q+1}) \ldots (z - \beta_n)$. Note that this shows $T(z)$ is finite for $z \in S$.

Let $z, z' \in S$. Then, both z/β_i and z'/β_i are infinitesimal for $i = q+1, \ldots, n$. From this we deduce that

$$\begin{aligned} \frac{T(z)}{T(z')} &= \frac{(z - \beta_{q+1}) \ldots (z - \beta_n)}{(z' - \beta_{q+1}) \ldots (z' - \beta_n)} \\ &\approx 1. \end{aligned}$$

So we get $T(z) \approx T(z')$ for all $z, z' \in S$, since $T(z')$ is finite. Let c be the common value of $\mathrm{st}(T(z))$ for $z \in S$. Since $p_0 \not\equiv 0$ (by assumption, some a_j is not infinitesimal), there is a standard $z \in \mathbb{C}$ that is distinct from $u_1, \ldots, u_q$ and is not a root of p_0. For that z, $T(z) \not\approx 0$. So $c \neq 0$.

For any $z \in \mathbb{C}$ different from $u_1, \ldots, u_q$, we get

$$\begin{aligned} p_0(z) &\approx T(z)(z-u_1)\ldots(z-u_q) \\ &\approx c(z-u_1)\ldots(z-u_q) \end{aligned}$$

Since $p_0(z)$ and $c(z-u_1)\ldots(z-u_q)$ are standard, this yields

$$p_0(z) = c(z-u_1)\ldots(z-u_q)$$

for all $z \in \mathbb{C} \setminus \{u_1, \ldots, u_q\}$ and hence for all $z \in \mathbb{C}$. □

7.2. **Corollary.** $\mathcal{R}_n \colon \mathcal{P}_n \to \mathcal{M}_n$ *is continuous.*

Proof. Take any standard $p_0 \in \mathcal{P}_n$ and $p \in {}^*\mathcal{P}_n$ in the monad of p_0. Since p_0 has degree n, the argument above shows that $p(z)$ has no infinite roots. Further, Theorem 7.1 yields that ${}^*\mathcal{R}_n(p) \approx \mathcal{R}_n(p_0)$ in ${}^*\mathcal{M}_n$. (See Example 5.29 for a discussion of the standard part map in ${}^*\mathcal{M}_n$.) Therefore $\mathcal{R}_n$ satisfies the nonstandard criterion for continuity at p_0. □

Vector spaces

7.3. **Definition.** Fix $n \geq 1$ and $0 \leq k \leq n$ and consider $\mathbb{C}^n$ as a vector space over $\mathbb{C}$. Denote by $G^{n,k}$ the set of all k-dimensional subspaces of $\mathbb{C}^n$.

The spaces $G^{n,k}$ are called *Grassmannians on* $\mathbb{C}$, once they are given a suitable topology. We examine $G^{n,k}$ as a topological space by focusing on what turns out to be its standard part map. By transfer, ${}^*(G^{n,k})$ is the set of all internal k-dimensional subspaces of ${}^*\mathbb{C}^n$, which we consider as an internal vector space over ${}^*\mathbb{C}$.

7.4. **Definition.** For $E \in {}^*G^{n,k}$ we define

$${}^\circ E = \{\mathrm{st}(v) \mid v \in E \text{ is finite in } {}^*\mathbb{C}^n\}.$$

It's easily checked that ${}^\circ E$ is a subspace of $\mathbb{C}^n$, considered as a vector space over $\mathbb{C}$. We want to determine its dimension.

7.5. **Lemma.** *If* $E \in {}^*G^{n,k}$*, then* $\dim_{\mathbb{C}}({}^\circ E) = k$.

Proof. We introduce the usual inner product $\langle \cdot, \cdot \rangle$ defined for $v, w \in \mathbb{C}^n$ by

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w}_i.$$

The nonstandard extension of this function will be written $\langle \cdot, \cdot \rangle$; it maps ${}^*\mathbb{C}^n$ to ${}^*\mathbb{C}$ and has all the formally expressible properties of the usual inner product. Since E is an internal subspace of ${}^*\mathbb{C}^n$, it has an orthonormal basis $v_1, \ldots, v_k$, by transfer. We have

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

In particular, $v_1, \ldots, v_n$ are finite in ${}^*\mathbb{C}^n$, so $\mathrm{st}(v_1), \ldots, \mathrm{st}(v_k) \in {}^\circ E$. It's easy to check that $\mathrm{st}(v_1), \ldots, \mathrm{st}(v_k)$ are orthonormal in $\mathbb{C}^n$, so they are independent. We complete the proof by showing that $\mathrm{st}(v_1), \ldots, \mathrm{st}(v_k)$ spans ${}^\circ E$ over $\mathbb{C}$. If $v \in E$, then $v = \alpha_1 v_1 + \cdots + \alpha_k v_k$ for some $\alpha_1, \ldots, \alpha_k \in {}^*\mathbb{C}$. Assume v is finite in ${}^*\mathbb{C}^n$, so $\mathrm{st}(v)$ is a typical element of ${}^\circ E$. Then $\langle v, v_j \rangle = \alpha_j$ for each j, so by the Cauchy-Schwarz inequality, each α_j is finite. The continuity of vector operations yields

$$\mathrm{st}(v) = \textstyle\sum_{j=1}^{k} \mathrm{st}(\alpha_j)\, \mathrm{st}(v_j) \in \mathrm{span}_{\mathbb{C}}(\mathrm{st}(v_1), \ldots, \mathrm{st}(v_k)).$$

□

7.6. **Exercise.** The Lemma shows that the map $E \mapsto {}^\circ E$ takes ${}^*G^{n,k}$ into $G^{n,k}$. Show that there is a topology on $G^{n,k}$ for which this map is the standard part map. Use obvious properties of this standard part map to show that in this topology, $G^{n,k}$ is a compact, Hausdorff space.

Finite dimensional linear operators

Consider $V = \mathbb{C}^n$ as a vector space over $\mathbb{C}$ of dimension n, and let $T\colon V \to V$ be a linear operator. Recall that the characteristic polynomial of T is $p(\lambda) = \det(T - \lambda I)$, where I is the identity on V. The roots of $p(\lambda)$ are the eigenvalues of T (i.e., those λ such that $(T - \lambda I)v = 0$ has a nonzero solution $v \in V$). Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T and let $s_1, \ldots, s_m$ be their multiplicities as roots of p, so we have

$$p(\lambda) = \prod_{j=1}^{m} (\lambda - \lambda_j)^{s_j}.$$

For $j = 1, \dots, m$, let

$$U_j = \{v \in V \mid (T - \lambda_j I)^{s_j} v = 0\}.$$

Then $U_1, \dots, U_m$ are subspaces of V, each of which is invariant under T. For each j, U_j is called the *(generalized) eigenspace of* T associated to the eigenvalue λ_j.

The first main structure theorem for linear operators on finite dimensional vector spaces over $\mathbb{C}$ is the following result:

7.7. **Theorem** (Primary Decomposition Theorem)**.** *In the situation described above, V equals the direct sum $U_1 \oplus \cdots \oplus U_m$ and $\dim U_j = s_j$ for all $j = 1, \dots, m$.*

We will prove the above theorem using a combination of standard and nonstandard techniques. Nonstandard analysis comes in especially at the end to aid proving that these eigenspaces have the right dimensions.

We first prove some standard facts from linear algebra.

7.8. **Lemma.** *Given a vector space V of dimension n over $\mathbb{C}$ and a linear operator $T\colon V \to V$, there exists a basis $v_1, \dots, v_n$ for V such that each subspace span$(v_1, \dots, v_i)$ is T-invariant. The eigenvalues of T can be listed (with multiplicities) as $\alpha_1, \dots, \alpha_n$ so that for each $i = 1, \dots, n$ one has $(T - \alpha_i I)v_i \in$ span$(v_1, \dots, v_{i-1})$.*

Proof. We prove the first part of the Lemma by induction on n. The lemma is trivial for $n = 0$. We assume the lemma holds for $n - 1 \geq 0$ and prove it for n. Fix an eigenvector $u \neq 0$ of T. Expand u to a basis $u, w_1, \dots, w_{n-1}$ for V. Let $U = \text{span}(u)$ and let $W = \text{span}(w_1, \dots, w_{n-1})$. Let $P\colon V \to W$ be the projection onto W with $\ker P = U$ and define $S\colon W \to W$ by $S = PT$. The lemma holds for S on W by induction, so let $z_1, \dots, z_{n-1}$ be such a basis for W. Then $u, z_1, \dots, z_{n-1}$ is a basis for V and it is a straightforward exercise to show that for each $i = 1, \dots, n-1$ the subspace $\text{span}(u, z_1, \dots, z_i)$ is T-invariant.

Now let $v_1, \dots, v_n$ be a basis for V such that each subspace $\text{span}(v_1, \dots, v_i)$ is T-invariant. For each i let β_i be the coefficient of v_i when $T(v_i)$ is written as a linear combination of $v_1, \dots, v_i$. (So $\beta_1, \dots, \beta_n$ are the diagonal entries of the matrix of T with respect to the basis $v_1, \dots, v_n$, which is an upper triangular matrix.) It is an easy exercise to show that $\beta_1, \ldots, \beta_n$ are the eigenvalues of T. □

One important consequence of this Lemma is the following, which we use later.

7.9. **Theorem** (Cayley-Hamilton Theorem)**.** *Let $T\colon V \to V$ have characteristic polynomial $p(\lambda) = \det(T - \lambda I) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$. Then $p(T) = a_0 + a_1 T + \cdots + a_n T^n = 0$.*

Proof. Let $v_1, \ldots, v_n$ be a basis of V obtained using Lemma 7.8. Let the eigenvalues of T be (with repetitions) $\alpha_1, \ldots, \alpha_n$. The operators $T - \alpha_i I$ for $i = 1, \ldots, n$ commute and the composition of all of them is equal to the operator $p(T)$. An easy argument shows that $p(T)v_i = 0$ for all i and hence $p(T) = 0$. □

Now we develop some ideas from nonstandard analysis that are needed for the proof of the Primary Decomposition Theorem.

We put the usual topologies on $V = \mathbb{C}^n$ and on the space of $n \times n$ matrices over $\mathbb{C}$. If $A = (a_{ij})$ and $B = (b_{ij})$, then $A \approx B$ if $a_{ij} \approx b_{ij}$ for all i, j. We say that A is finite if a_{ij} is finite for all i, j; these are the nearstandard matrices relative to this topology. We also topologize the space of all linear operators on V in the usual way. Given internal *linear operators $T, S\colon {}^*V \to {}^*V$, we have $T \approx S$ if and only if $Tx \approx Sx$ for all finite $x \in {}^*V$. We say that T is finite if Tx is finite for all finite $x \in {}^*V$; these are the nearstandard operators on ${}^*V$ with respect to this topology.

7.10. **Proposition.** *If the internal *linear operators $T, S\colon {}^*V \to {}^*V$ are finite, and $T \approx S$, then one can list the eigenvalues of S and T as $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, such that $\alpha_i \approx \beta_i$ for $i = 1, \ldots, n$.*

Proof. Let A, B be the matrices of T, S with respect to the standard basis of ${}^*\mathbb{C}^n$. Check that $A \approx B$. Then the polynomials $\det(T - \lambda I)$ and $\det(S - \lambda I)$ have coefficients that differ by an infinitesimal, degree by degree, since these are given by continuous functions of the entries of the representing matrices. The result then follows by Theorem 7.1. □

7.11. **Proposition.** *Given any $T\colon V \to V$, there exists $S\colon {}^*V \to {}^*V$ such that $S \approx {}^*T$ and S has n distinct eigenvalues (in ${}^*\mathbb{C}$). Equivalently, given an $n \times n$ matrix A over $\mathbb{C}$, there exists an $n \times n$ matrix B over ${}^*\mathbb{C}$ such that $B \approx A$ and B has n distinct eigenvalues (in ${}^*\mathbb{C}$).*

Proof. Use Lemma 7.8 to get an upper triangular matrix U and an invertible matrix D over $\mathbb{C}$ such that $A = DUD^{-1}$. Let U' be a perturbation of U (i.e., satisfying $U' \approx U$) obtained by perturbing the diagonal entries of U to make them distinct. Let $B = DU'D^{-1}$. Since $U' \approx U$ and matrix multiplication is continuous, it follows that $B \approx A$. The eigenvalues of B equal the eigenvalues of U', which are distinct. □

We now give the proof of the Primary Decomposition Theorem. The first part does not use nonstandard analysis.

Part I, proof of Theorem 7.7; standard arguments.
We prove here that $V = U_1 \oplus \cdots \oplus U_m$. Let $p_j(\lambda) = \frac{p(\lambda)}{(\lambda-\lambda_j)^{s_j}}$. Note that $p_1, \ldots, p_m$ have no common root. Since $\mathbb{C}[\lambda]$ is a PID, there exists polynomials $q_1, \ldots, q_m \in \mathbb{C}[\lambda]$ such that $\sum_{j=1}^m q_j p_j = 1$. For each $j = 1, \ldots, m$, let $E_j = q_j(T)p_j(T)$. We observe that $E_1 + \cdots + E_m = Id_V$ and $E_iE_j = 0$ whenever $i \neq j$ (using Theorem 7.9 since p_ip_j is divisible by p in $\mathbb{C}[\lambda]$). Combining these, it follows that $E_j^2 = E_j$ for all j. Hence we get that V is the direct sum of the subspaces $E_1V, \ldots, E_mV$.

To finish this part of the proof, we need to show that $E_jV = U_j$ for all j. Suppose that $u \in E_jV$ so that $u = E_jv$ for some $v \in V$. Then $(T - \lambda_jI)^{s_j}u = (T - \lambda_jI)^{s_j}E_jv = q_j(T)[(T - \lambda_jI)^{s_j}p_j(T)]v = q_i(T)p(T)v = 0$, so $u \in U_j$. Hence $E_jV \subseteq U_j$.

For the opposite containment, suppose $u \in U_j$. For any $i \neq j$, p_i is divisible by $(\lambda - \lambda_j)^{s_j}$, so $E_iu = 0$. Therefore, $u = (E_1 + \cdots + E_n)u = E_iu + \cdots + E_nu = E_iu \in E_jV$. It follows that $U_j \subseteq E_jV$. Hence $E_jV = U_j$ for all j and we are done with this part of the proof. □

We now complete the proof of the Primary Decomposition Theorem using nonstandard analysis. It remains to show that $\dim U_j = s_j$ for $j = 1, 2, \ldots, m$.

Part II, proof of Theorem 7.7; nonstandard arguments.
Use Proposition 7.11 to get $S\colon {}^*V \to {}^*V$ such that $S \approx {}^*T$ and S has distinct eigenvalues. Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of S, and let $F = \{\alpha_1, \ldots, \alpha_n\}$. Observe that the polynomials $\det({}^*T - \lambda I)$ and $\det(S - \lambda I)$ over ${}^*\mathbb{C}$ have coefficients that differ by an infinitesimal, degree by degree, and both have leading coefficient± 1. By Theorem 7.1, we can order the eigenvalues of S so that they are infinitely close to the eigenvalues of T listed with multiplicity. Hence the standard part map takes F onto $\{\lambda_1, \ldots, \lambda_m$ and, moreover, for each $j = 1, \ldots, m$ we have that s_j is the number of i for which $\alpha_i \approx \lambda_j$.

Let $v_1, \ldots, v_n$ be eigenvectors for S, satisfying $Sv_i = \alpha_i v_i$ for all $i = 1, \ldots, n$; thus $v_1, \ldots, v_n$ is a basis for ${}^*V$ over ${}^*\mathbb{C}$, since $\alpha_1, \ldots, \alpha_n$ are distinct. Let $W_j = \mathrm{span}_{{}^*\mathbb{C}}\{v_i \mid \alpha_i \approx \lambda_j\}$, so $\dim_{{}^*\mathbb{C}} W_j = s_j$. Lemma 7.5 yields $\dim_{\mathbb{C}} {}^\circ W_j = s_j$. We will now prove that ${}^\circ W_j \subseteq U_j$ (and hence $\dim_{\mathbb{C}} U_j \geq s_j$) for all $j = 1, \ldots, m$.

Let $w \in W_j$ be finite. If $\beta_1, \ldots, \beta_{s_j}$ is an enumeration of the α_i satisfying $\alpha_i \approx \lambda_j$, then we have

$$(T - \lambda_j I)^{s_j}\,\mathrm{st}(w) \approx (S - \beta_1 I) \ldots (S - \beta_{s_j} I)w = 0,$$

so $(T - \lambda_j I)^{s_j}\,\mathrm{st}(w) = 0$. It follows that ${}^\circ W_j \subseteq U_j$.

From $V = U_1 \oplus \cdots \oplus U_m$ it follows that $\dim U_j = s_j$, completing the proof of the Primary Decomposition Theorem. □

This result immediately yields another of the standard normal form theorems for finite dimensional operators:

7.12. **Exercise.** We use the notation of the Primary Decomposition Theorem. Show that $T = D + N$ where D is diagonalizable, N is nilpotent, and $DN = ND$. (Hint: Let T_j be the restriction of T to U_j, for each $j = 1, \ldots, m$. Let $D_j = \lambda_j I_{U_j}$ and $N_j = T_j - D_j$. Show that N_j is nilpotent, and $D_j N_j = N_j D_j$. Note that T is the direct sum of the T_j's.)

We close this section by briefly indicating the continuity of the primary decomposition as the operator T on V varies. We do this quickly from a nonstandard analysis point of view, without being careful about the topology on “primary decompositions.”. Let $T\colon V \to V$ be any standard linear operator on $V = \mathbb{C}^n$ and let $S\colon {}^*V \to {}^*V$ satisfy $S \approx {}^*T$. Let ${}^*V = W_1 \oplus \cdots \oplus W_k$ be the internal primary decomposition of S; let $\alpha_1, \ldots, \alpha_k$ be the distinct eigenvalues of S, arranged so that W_i is the eigenspace of S associated to α_i for each i. Let r_i be the multiplicity of α_i as an eigenvalue of S, so $\dim_{{}^*\mathbb{C}}(W_i) = r_i$ for each i, by the internal version of the primary decomposition theorem.

Suppose the eigenvalues of T are, as above, $\lambda_1, \ldots, \lambda_m$ with multiplicities $s_1, \ldots, s_m$, and let $V = U_1 \oplus \cdots \oplus U_m$ be the corresponding primary decomposition of T. Since $S \approx {}^*T$, we know that for each $j = 1, \ldots, m$ we have

$$s_j = \sum \{r_i \mid \alpha_i \approx \lambda_j\}.$$

For each j let

$$V_j = \bigoplus \{W_i \mid \alpha_i \approx \lambda_j\}.$$

(Here we work over ${}^*\mathbb{C}$.)

7.13. **Exercise.** Show that $U_j = {}^\circ V_j$ for each $j = 1, \ldots, m$.

This suggests that the primary decomposition does indeed depend continuously on the operator being decomposed. (Exercise: make the standard content of this statement precise. That is, give a topology on the space of primary decompositions such that, using the notation above, the standard part of $(W_1, \ldots, W_k)$ is $({}^\circ V_1, \ldots, {}^\circ V_m)$.)

The following example looks slightly below the surface of this comparison of primary decompositions:

7.14. **Example** (n=2)**.** Let $V = \mathbb{C}^2$ and let A and B be the following matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \delta \\ 0 & 1+\epsilon \end{pmatrix}$$

where $\delta, \epsilon \approx 0$ and $\frac{\delta}{\epsilon}$ is infinite. Then $A \approx B$. A basis of eigenvectors for B is given by $(1,0)$ and $(1, \frac{\epsilon}{\delta}) \approx (1,0)$. Hence, the primary decomposition for (the operator associated to) B is given by $W_1 = \operatorname{span}_{{}^*\mathbb{C}}(1,0)$ and $W_2 = \operatorname{span}_{{}^*\mathbb{C}}(1, \frac{\epsilon}{\delta})$. The canonical projections of ${}^*\mathbb{C}^2 = W_1 \oplus W_2$ onto W_1 and W_2 are given by the matrices

$$E_1 = \begin{pmatrix} 0 & -\frac{\delta}{\epsilon} \\ 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & \frac{\delta}{\epsilon} \\ 0 & 0 \end{pmatrix}$$

Note that the matrices E_1 and E_2 fail to be nearstandard. However, the projection $E_1 \oplus E_2$ is nearstandard and its standard part as an operator is indeed the projection associated to the primary decomposition of the operator (associated to) A.