

8. MEASURE THEORY

We begin the development of measure theoretic tools that have become central to many applications of nonstandard analysis. In this section we develop the measure theory and in the next section we cover integration. For things to work smoothly, we assume that our nonstandard extension satisfies at least ω_1 -saturation.

The starting point of this development is an *internal measure space* $(\Omega, \mathcal{A}, \mu)$. Here Ω is an internal set, \mathcal{A} is an internal Boolean subalgebra of ${}^*\mathcal{P}(\Omega)$, and $\mu: \mathcal{A} \rightarrow {}^*\mathbb{R}$ is an internal, nonnegative, *finitely* additive function.

We define ${}^\circ\mu: \mathcal{A} \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$ by

$${}^\circ\mu(A) = \begin{cases} \text{st}(\mu(A)) & \text{if } \mu(A) \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

It is easily checked that ${}^\circ\mu$ is a finitely additive measure on \mathcal{A} in the usual standard sense.

8.1. Definition. Let $X \subseteq \Omega$ be any set (possibly external). We call X a *${}^\circ\mu$ -null set* if, for all standard $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $X \subseteq A$ and ${}^\circ\mu(A) < \epsilon$.

8.2. Lemma. *The collection of ${}^\circ\mu$ -null sets is closed under countable unions.*

Proof. Suppose $(X_n \mid n \in \mathbb{N})$ is a sequence of ${}^\circ\mu$ -null sets, and $X = \bigcup_{n \in \mathbb{N}} X_n$. Fix a standard $\epsilon > 0$. For each n , choose $B_n \in \mathcal{A}$ with $X_n \subseteq B_n$ and ${}^\circ\mu(B_n) < \epsilon 2^{-(n+1)}$. For each k , we have $X_0 \cup \dots \cup X_k \subseteq B_0 \cup \dots \cup B_k$ and $\mu(B_0 \cup \dots \cup B_k) \leq \mu(B_0) + \dots + \mu(B_k) < \epsilon$. Use ω_1 -saturation to obtain $B \in \mathcal{A}$ such that $B_k \subseteq B$ for every k and $\mu(B) < \epsilon$. Since $X_k \subseteq B_k \subseteq B$ for every k , it follows that $X \subseteq B$. \square

8.3. Remark. Suppose $X \subseteq \Omega$ and let A_1, A_2 be sets in \mathcal{A} such that $A_1 \triangle X$ and $A_2 \triangle X$ are ${}^\circ\mu$ -null sets. It is easy to check that ${}^\circ\mu(A_1) = {}^\circ\mu(A_2)$. Indeed, $A_1 \triangle A_2$ is contained in $(A_1 \triangle X) \cup (A_2 \triangle X)$, which is a ${}^\circ\mu$ -null set; thus $\mu(A_1 \triangle A_2)$ is infinitesimal.

Our first main objective is to show that ${}^\circ\mu$ extends to a σ -additive measure on the σ -algebra of subsets of Ω that is generated by \mathcal{A} ; we denote this

σ -algebra by $\sigma(\mathcal{A})$. We could apply the Carathéodory Extension Theorem to prove the existence of this extension. However, in this nonstandard situation the construction of the extension is much easier than it is in general, so we give the details in full. To prove this extension result, we first deal with finite measure spaces and then take up the general situation.

Until further notice, $(\Omega, \mathcal{A}, \mu)$ is an internal measure space *with* $\mu(\Omega)$ *finite*.

8.4. Definition.

- (1) A set $X \subseteq \Omega$ is *strongly* ${}^\circ\mu$ -*measurable* if there exists $A \in \mathcal{A}$ such that $A\Delta X$ is a ${}^\circ\mu$ -null set.
- (2) A set $X \subseteq \Omega$ is ${}^\circ\mu$ -*measurable* if, for all standard $\epsilon > 0$, there exist $B, C \in \mathcal{A}$ with $B \subseteq X \subseteq C$ and $\mu(C \setminus B) < \epsilon$.
- (3) \mathcal{A}_L is the collection of strongly ${}^\circ\mu$ -measurable subsets of Ω .
- (4) For $X \in \mathcal{A}_L$, we define $\mu_L(X) = {}^\circ\mu(A)$, where A is any element of \mathcal{A} such that $A\Delta X$ is a ${}^\circ\mu$ -null set. (This is well-defined by the preceding remark.)

8.5. Lemma. *The following are equivalent for any $X \subseteq \Omega$:*

- (1) *X is strongly ${}^\circ\mu$ -measurable.*
- (2) *X is ${}^\circ\mu$ -measurable.*

Proof. (2 \Rightarrow 1) For each $n \geq 1$, choose $B_n, C_n \in \mathcal{A}$ with $B_n \subseteq X \subseteq C_n$ and $\mu(C_n \setminus B_n) < \frac{1}{n}$. Without loss of generality, take $(B_n \mid n \geq 1)$ to be an increasing family and take $(C_n \mid n \geq 1)$ to be decreasing. Use ω_1 -saturation to obtain $A \in \mathcal{A}$ such that $B_n \subseteq A \subseteq C_n$ for all $n \geq 1$. Then $A\Delta X \subseteq C_n \setminus B_n$ for all $n \geq 1$, and hence $A\Delta X$ is a ${}^\circ\mu$ -null set.

(1 \Rightarrow 2) Suppose that $A \in \mathcal{A}$ is such that $A\Delta X$ is a ${}^\circ\mu$ -null set. Take a standard $\epsilon > 0$ and choose $B \in \mathcal{A}$ with $A\Delta X \subseteq B$ and $\mu(B) < \epsilon$. Then $A \setminus B \subseteq X \subseteq A \cup B$, and $\mu((A \cup B) \setminus (A \setminus B)) = \mu(B) < \epsilon$. \square

8.6. Theorem. *\mathcal{A}_L is a σ -algebra of subsets of Ω containing \mathcal{A} and μ_L is a σ -additive measure on \mathcal{A}_L extending ${}^\circ\mu$.*

Proof. It is clear that \mathcal{A}_L contains \mathcal{A} , that μ_L extends ${}^\circ\mu$, that \mathcal{A}_L is closed under finite Boolean operations, and that μ_L is finitely additive on \mathcal{A}_L . It remains to show that \mathcal{A}_L is closed under countable unions and that μ_L is countably additive.

Suppose $(X_n \mid n \in \mathbb{N})$ are elements of \mathcal{A}_L . We need to show that $\bigcup_{n \in \mathbb{N}} X_n$ is in \mathcal{A}_L and that $\mu_L(\bigcup_{n \in \mathbb{N}} X_n)$ is the limit of $\mu_L(X_0 \cup \dots \cup X_n)$ as n tends to ∞ .

From the definition of \mathcal{A}_L , there exist $(A_n \mid n \in \mathbb{N})$ in \mathcal{A} such that $A_n \Delta X_n$ is a ${}^\circ\mu$ -null set for all $n \in \mathbb{N}$. Note that

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \Delta \left(\bigcup_{n \in \mathbb{N}} X_n\right) \subseteq \bigcup_{n \in \mathbb{N}} (A_n \Delta X_n)$$

and the set on the right is a ${}^\circ\mu$ -null set by Lemma 8.2. It follows that

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \Delta \left(\bigcup_{n \in \mathbb{N}} X_n\right)$$

is a ${}^\circ\mu$ -null set.

This reduces our problem to showing that $\bigcup_{n \in \mathbb{N}} A_n$ is in \mathcal{A}_L and that $\mu_L(\bigcup_{n \in \mathbb{N}} A_n)$ is the limit of $\mu_L(A_0 \cup \dots \cup A_n) = {}^\circ\mu(A_0 \cup A_2 \cup \dots \cup A_n)$ as n tends to ∞ .

Let $\alpha = \lim_n {}^\circ\mu(A_0 \cup A_2 \cup \dots \cup A_n)$, which equals $\sup_n {}^\circ\mu(A_0 \cup A_2 \cup \dots \cup A_n)$. Use ω_1 -saturation to get $A \in \mathcal{A}$ such that $A_0 \cup A_2 \cup \dots \cup A_n \subseteq A$ and $\mu(A) < \alpha + \frac{1}{n}$ for all n . One easily shows that $A \Delta (\bigcup_{n \in \mathbb{N}} A_n)$ is a ${}^\circ\mu$ -null set and ${}^\circ\mu(A) = \alpha$. It follows that $\mu_L(\bigcup_{n \in \mathbb{N}} X_n) = \mu_L(\bigcup_{n \in \mathbb{N}} A_n) = {}^\circ\mu(A) = \alpha$. \square

8.7. Corollary. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space with $\mu(\Omega)$ finite. Then ${}^\circ\mu$ has a unique extension ν to a σ -additive measure on $\sigma(\mathcal{A})$. The completion of the measure space $(\Omega, \sigma(\mathcal{A}), \nu)$ is $(\Omega, \mathcal{A}_L, \mu_L)$. (That is, \mathcal{A}_L is exactly the result of adding all μ_L -null subsets of Ω to $\sigma(\mathcal{A})$.)*

Proof. This is immediate from the definitions of \mathcal{A}_L and μ_L , and the previous Theorem. \square

8.8. Examples. We give two important examples of internal measure spaces.

(1) Let (T, \mathcal{B}, ν) be a standard measure space in $U(X)$ and take $(\Omega, \mathcal{A}, \mu)$ to be the nonstandard extension $({}^*T, {}^*\mathcal{B}, {}^*\nu)$. The map $S \mapsto {}^*S$ is a measure preserving isomorphism of (T, \mathcal{B}, ν) into $({}^*T, ({}^*\mathcal{B})_L, ({}^*\nu)_L)$. Note that $({}^*\nu)_L$ is σ -additive on $({}^*\mathcal{B})_L$, even if the original measure space (T, \mathcal{B}, ν) is only finitely additive.

(2) Let Ω be a nonempty hyperfinite set and let \mathcal{A} be the Boolean algebra of all internal subsets of Ω . For each $A \in \mathcal{A}$ let $|A|$ be the internal cardinality

of A , an element of \mathbb{N} . Define μ on \mathcal{A} by

$$\mu(A) = \frac{|A|}{|\Omega|}$$

for all $A \in \mathcal{A}$. Note that $(\Omega, \mathcal{A}, \mu)$ is an internal finitely additive probability space ($\mu(\Omega) = 1$). The measure space $(\Omega, \mathcal{A}_L, \mu_L)$ is called a *counting measure*. Later in this section we will see that we can use these measures to construct Lebesgue measure and other important measures from analysis in a direct and simple manner.

8.9. Terminology. Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space for which $\mu(\Omega)$ is finite. The measure space $(\Omega, \mathcal{A}_L, \mu_L)$ is called the *Loeb measure space* induced by $(\Omega, \mathcal{A}, \mu)$. This construction was introduced by Peter Loeb in a 1972 paper.

The following result shows that intersections of *uncountable* families of sets from \mathcal{A} are ${}^\circ\mu$ -measurable, as long as our nonstandard extension is sufficiently saturated.

8.10. Lemma. (*Assume κ -saturation.*)

For any $\{A_i \mid i \in I\} \subseteq \mathcal{A}$ with $\text{card } I < \kappa$, the set $\bigcap\{A_i \mid i \in I\}$ is ${}^\circ\mu$ -measurable and

$$\mu_L\left(\bigcap\{A_i \mid i \in I\}\right) = \inf\{\mu(A_{i_1} \cap \cdots \cap A_{i_k}) \mid k \geq 1, i_1, \dots, i_k \in I\}.$$

Proof. Set $X = \bigcap\{A_i \mid i \in I\}$. Let $J = \{\alpha \subseteq I \mid \alpha \text{ is finite}\}$, and for $\alpha \in J$, define $A_\alpha = \bigcap\{A_i \mid i \in \alpha\}$. Then $X = \bigcap\{A_\alpha \mid \alpha \in J\}$. Define $r = \inf\{\mu(A_\alpha) \mid \alpha \in J\}$. Then $r \in \mathbb{R}$ and $r \leq {}^\circ\mu(\Omega)$. We claim that $X \in \mathcal{A}_L$ and $\mu_L(X) = r$. To prove this, fix $\varepsilon > 0$ standard. Use κ -saturation to produce a set $A \in \mathcal{A}$ such that $A \subseteq X$ and $\mu(A) > r - \varepsilon$. Equivalently, we need $A \in \mathcal{A}$ such that $A \subseteq A_i$ for all $i \in I$ and $\mu(A) > r - \varepsilon$. These conditions are finitely satisfied by selecting A suitably from among the A_α 's. \square

We now consider the case of infinite measures. That is, we still take $(\Omega, \mathcal{A}, \mu)$ to be an internal measure space, but we drop the requirement that $\mu(\Omega)$ be a finite element of ${}^*\mathbb{R}$. Our objective is still to show that ${}^\circ\mu$ extends to a σ -additive measure on $\sigma(\mathcal{A})$.

8.11. **Definition.** A set $X \subseteq \Omega$ is *countably determined* if there is a countable family $(A_n \mid n \in \mathbb{N})$ in \mathcal{A} such that X is in the complete Boolean algebra of subsets of Ω generated by $(A_n \mid n \in \mathbb{N})$, which we denote by $\mathcal{B}(A_n \mid n \in \mathbb{N})$.

8.12. **Remark.** Every set in $\sigma(\mathcal{A})$ is countably determined. Indeed, for any set X in \mathcal{A} there exists a countable Boolean subalgebra \mathcal{A}_0 of \mathcal{A} such that $X \in \sigma(\mathcal{A}_0) \subseteq \mathcal{B}(\mathcal{A}_0)$. The proof is by induction on the construction of X as an element of $\sigma(\mathcal{A})$.

Without loss of generality, when we consider a countably determined set X , we may assume that the family $(A_n \mid n \in \mathbb{N})$ for which $X \in \mathcal{B}(A_n \mid n \in \mathbb{N})$ is a Boolean subalgebra of ${}^*\mathcal{P}(\Omega)$. Otherwise we can close the given family of sets under finite Boolean operations and still have a countable family.

8.13. **Exercise.** Let $\mathcal{X} = \{\bigcap_{n \in \mathbb{N}} A_n^{\varepsilon_n} \mid \varepsilon: \mathbb{N} \rightarrow \{-1, +1\}\}$, where $A^{+1} = A$ and $A^{-1} = \Omega \setminus A$. This family of sets is contained in $\mathcal{B}(A_n \mid n \in \mathbb{N})$ and it partitions Ω . Prove that $\mathcal{B}(A_n \mid n \in \mathbb{N})$ is precisely the family of all possible unions of sets from \mathcal{X} .

8.14. **Proposition.** Let $(A_n \mid n \in \mathbb{N}) \subseteq \mathcal{A}$ and $X \in \mathcal{B}(A_n \mid n \in \mathbb{N})$. Then exactly one of the following holds:

- (1) There exists $A \in \mathcal{A}$ such that $A \subseteq X$ and $\mu(A)$ is infinite.
- (2) There exists a family $(B_n \mid n \in \mathbb{N}) \subseteq \mathcal{A}$ such that each $\mu(B_n)$ is finite and $X \subseteq \bigcup (B_n \mid n \in \mathbb{N})$.

Proof. We prove first that if (2) fails, then (1) must hold. We assume that $\{A_n \mid n \in \mathbb{N}\}$ is closed under finite Boolean operations.

Let $S = \{A_n \mid \mu(A_n) \text{ is finite}\}$. If S covers X then the second condition holds. Thus S does not cover X , so there exists $a \in X \setminus \bigcup S$. Therefore, for each $n \in \mathbb{N}$, if $a \in A_n$, then $\mu(A_n)$ is infinite. For each $n \in \mathbb{N}$ define

$$\varepsilon_n = \begin{cases} +1 & \text{if } a \in A_n \\ -1 & \text{if } a \notin A_n \end{cases}$$

and consider $Y = \bigcap_{n \in \mathbb{N}} A_n^{\varepsilon_n}$.

Evidently $a \in Y$. Therefore, by Exercise 8.13, $Y \subseteq X$. Using ω_1 -saturation we can show there exists $A \in \mathcal{A}$ with $\mu(A)$ infinite, and $A \subseteq Y$, which will prove (1). To apply ω_1 -saturation in this way, we note that for each $n \in \mathbb{N}$ the internal set $A = A_0^{\varepsilon_0} \cap \dots \cap A_n^{\varepsilon_n}$ satisfies $\mu(A) \geq n$.

Finally, suppose both conditions (1) and (2) hold. So we have A and B_n for $n \in \mathbb{N}$, all from \mathcal{A} , such that $\mu(A)$ is infinite, each $\mu(B_n)$ is finite, and $A \subseteq X \subseteq \bigcup(B_n \mid n \in \mathbb{N})$. By ω_1 -saturation, there exists $k \in \mathbb{N}$ such that $A \subseteq B_0 \cup \dots \cup B_k$, which is an obvious contradiction. \square

The next result, when combined with our discussion of the Loeb measure in the finite case, provides a complete picture of the Loeb measure even in the case where $\mu(\Omega)$ is infinite: given a set X in the σ -algebra generated by \mathcal{A} , either the measure of X is forced to be infinite in an obvious way or X lies inside an internal set of finite measure; in the second case, there is set in \mathcal{A} from which X differs by a ${}^\circ\mu$ -null set.

8.15. Theorem. *Let $X \in \sigma(\mathcal{A})$. Exactly one of the following conditions holds:*

- (1) *For every $n \geq 1$ there exists $A \in \mathcal{A}$ with $A \subseteq X$ and $\mu(A) \geq n$.*
- (2) *There exists $B \in \mathcal{A}$ with $X \subseteq B$ and $\mu(B)$ finite; hence there is set in \mathcal{A} from which X differs by a ${}^\circ\mu$ -null set.*

Proof. We use proposition 8.14. If 8.14(1) holds, then clearly (1) of the present theorem is true. Otherwise, 8.14(2) holds; that is, there exist $(B_n \mid n \in \mathbb{N}) \subseteq \mathcal{A}$ with $X \subseteq \bigcup_{n \in \mathbb{N}} B_n$ and every $\mu(B_n)$ finite. Suppose that $\{\mu(B_0 \cup \dots \cup B_n) \mid n \in \mathbb{N}\}$ has a finite upper bound. Then an ω_1 -saturation argument produces $B \in \mathcal{A}$ with $\mu(B)$ finite and $B_n \subseteq B$ for all $n \in \mathbb{N}$, witnessing that (2) of the present result holds; for the second part of (2) we use our previous results for finite internal measure spaces applied to the restriction of $(\Omega, \mathcal{A}, \mu)$ to B .

So, arguing by contradiction, assume $\{\mu(B_0 \cup \dots \cup B_n) \mid n \in \mathbb{N}\}$ has no finite upper bound. Applying the previous results for finite measures, we obtain B'_n satisfying $B'_n \subseteq B_n$, $B'_n \in \mathcal{A}$, and $X \cap B_n \subseteq B'_n$, and such that there exists $A_n \in \mathcal{A}$ with $A_n \subseteq X \cap B_n$ and $\mu(B'_n \setminus A_n) < 2^{-n}$.

By the same reasoning as applied above to $\{B_n \mid n \in \mathbb{N}\}$, we may assume that $\{\mu(B'_1 \cup \dots \cup B'_n) \mid n \in \mathbb{N}\}$ has no finite upper bound. Then $\{\mu(A_1 \cup \dots \cup A_n) \mid n \in \mathbb{N}\}$ is also unbounded, meaning that the sets $A_1 \cup \dots \cup A_n \subseteq X$ witness that (1) of the present result holds. \square

8.16. Corollary. *${}^\circ\mu$ on \mathcal{A} extends uniquely to a σ -additive measure on $\sigma(\mathcal{A})$.*

Proof. For any $X \in \sigma(\mathcal{A})$, consider the alternatives in the previous Theorem. If (1) holds, then any extension of ${}^\circ\mu$ must give X infinite measure. If (2) holds, then there exists $A \in \mathcal{A}$ such that $A \Delta X$ is a ${}^\circ\mu$ -null set and $\mu(A)$ is finite. Then any extension of ${}^\circ\mu$ must assign ${}^\circ\mu(A)$ as the measure of X .

Let $\nu: \sigma(\mathcal{A}) \rightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$ defined as in the previous paragraph. Obviously ν extends ${}^\circ\mu$. It is easy to check using previous results in this section that ν is σ -additive; we leave this as an exercise. \square

PUSHING DOWN MEASURES

Consider a measure space (X, \mathcal{B}, ν) ; that is, \mathcal{B} is a σ -algebra on X and ν is a σ -additive measure. Let $\pi: X \rightarrow Y$ be surjective. A standard construction allows us to *push down* the measure space (X, \mathcal{B}, ν) onto Y via π . To do this, define \mathcal{B}_π to be the collection of subsets S of Y such that $\pi^{-1}(S) \in \mathcal{B}$. It is obvious that \mathcal{B}_π is a σ -algebra on Y . Moreover, we may define a measure ν_π on \mathcal{B}_π by $\nu_\pi(S) = \nu(\pi^{-1}(S))$. It's easy to check that ν_π is σ -additive. (Note that the map $S \mapsto \text{st}^{-1}(S)$ is a completely additive Boolean embedding from $\mathcal{P}(Y)$ into $\mathcal{P}(X)$.)

We'll refer to $(Y, \mathcal{B}_\pi, \nu_\pi)$ as the measure space obtained by *pushing down* (X, \mathcal{B}, ν) *onto* Y *along* π and we will also denote it as $(X, \mathcal{B}, \nu)_\pi$.

The main purpose of this subsection is to show that Lebesgue measure on $[0, 1]$ and the Haar measure on any compact topological group can be obtained by pushing down a Loeb counting measure on a hyperfinite set.

LEBESGUE MEASURE ON $[0, 1]$

Let $\Omega = \{\frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N}\} \subseteq {}^*[0, 1]$ and $\mathcal{A} = {}^*\mathcal{P}(\Omega)$, where N is an infinite element of ${}^*\mathbb{N}$. Define the normalized counting measure μ as usual by $\mu(A) = \frac{|A|}{N}$ for all $A \in \mathcal{A}$. This yields the Loeb measure space $(\Omega, \mathcal{A}_L, \mu_L)$. We push this down onto the interval $[0, 1]$ using the standard part map restricted to Ω , which is surjective since N is infinite.

8.17. Theorem. $(\Omega, \mathcal{A}_L, \mu_L)_{\text{st}}$ is the Lebesgue measure space on $[0, 1]$.

Proof. In the two Lemmas that follow, we prove the following facts: (1) The σ -algebra $(\mathcal{A}_L)_{\text{st}}$ contains every interval $(a, b) \subseteq [0, 1]$ and $(\mu_L)_{\text{st}}((a, b)) =$

$b - a$; hence it contains every open subset O of $[0, 1]$, and O is assigned the same measure by $(\mu_L)_{\text{st}}$ as by Lebesgue measure. (Namely, the measure of O in both cases is the convergent sum of the lengths of the maximal open intervals contained in O .) Hence the same is true of every closed subset of $[0, 1]$, by taking complements. (2) The measure $(\mu_L)_{\text{st}}$ is ‘inner regular’ on sets in $(\mathcal{A}_L)_{\text{st}}$. From these two facts it follows by an easy argument that $(\mathcal{A}_L)_{\text{st}}$ is exactly the collection of Lebesgue measurable subsets of $[0, 1]$ and $(\mu_L)_{\text{st}}$ is Lebesgue measure. \square

8.18. Lemma. *If $0 \leq a < b \leq 1$ and $S = (a, b)$, then $\text{st}^{-1}(S) \in \mathcal{A}_L$ and $\mu_L(\text{st}^{-1}(S)) = b - a$.*

Proof. Note that $\text{st}^{-1}(S) = \bigcap_{n \in \mathbb{N}}^* (a - \frac{1}{n}, b + \frac{1}{n}) \cap \Omega$, and hence $\text{st}^{-1}(S) \in \mathcal{A}_L$. An easy counting argument shows that $\mu_L(\text{st}^{-1}(S)) = \lim_{n \rightarrow \infty} (b - a + \frac{2}{n}) = b - a$. \square

8.19. Lemma. *Consider $S \subseteq [0, 1]$. If $\text{st}^{-1}(S) \in \mathcal{A}_L$, then for every standard $\epsilon > 0$ there is a closed set $C \subseteq [0, 1]$ such that $C \subseteq S$ and $\mu_L(\text{st}^{-1}(S \setminus C)) < \epsilon$.*

Proof. From the assumptions we get internal sets A and B such that $A \subseteq \text{st}^{-1}(S) \subseteq B$ and $\mu(B \setminus A) < \epsilon$. Taking standard part, we get $\text{st}(A) \subseteq S \subseteq \text{st}(B)$. Let $C = \text{st}(A)$, so C is closed and thus the Lebesgue measure of C is $\mu_L(\text{st}^{-1}(C))$. Also,

$$\mu_L(\text{st}^{-1}(S)) \geq \mu_L(\text{st}^{-1}(C)) \geq {}^\circ\mu(A) \geq \mu_L(\text{st}^{-1}(S)) - \epsilon \quad \square$$

HAAR MEASURE ON A COMPACT GROUP

A topological group is a group (G, \cdot) with a Hausdorff topology τ , such that the functions $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are τ -continuous.

A (left) Haar measure on (G, \cdot) is a Borel measure μ on G such that:

- (1) if $V \subseteq G$ is open and nonempty, then $\mu(V) > 0$.
- (2) if $g \in G$, there is an open neighborhood V of g such that $\mu(V) < \infty$.
- (3) if $B \subseteq G$ is Borel and $g \in G$, then $\mu(gB) = \mu(B)$ (left invariance of μ).

In the rest of this subsection we take (G, \cdot) to be a compact topological group, and we construct a left Haar measure on G by pushing down a Loeb counting measure on a hyperfinite set along the standard part map. (It’s

known that a left Haar measure μ on a compact group G is uniquely determined by the number $\mu(G)$; moreover, a left Haar measure is automatically right invariant. See *Measure Theory* by Paul Halmos, chapter XI. We do not prove these facts here.)

We assume that G is in $U(X)$ and that our nonstandard extension is κ -saturated, where κ is greater than the number of open sets in G .

8.20. Theorem. *There is a hyperfinite set $\Omega \subseteq {}^*G$ such that a left Haar measure on G is obtained by pushing down the Loeb probability space $(\Omega, {}^*\mathcal{P}(\Omega)_L, \mu_L)$ along the standard part map $\text{st}: \Omega \rightarrow G$, where μ is the normalized counting measure defined by $\mu(A) = \frac{|A|}{|\Omega|}$ for all internal $A \subseteq \Omega$.*

Proof. Using Lemma 5.7) we may take U to be an infinitesimal * neighborhood of the identity in *G . That is, U is * open, $e \in U$, and $U \subseteq \mu(e)$.

Consider the internal * open cover $\{gU \mid g \in {}^*G\}$. Since *G is * compact, by transfer this has a subcover $\{gU \mid g \in \Omega\}$, where $\Omega \subseteq {}^*G$ is hyperfinite. The key step in this construction is to choose Ω so that $\{gU \mid g \in \Omega\}$ covers *G and $|\Omega|$ is the least possible. This can be done because the condition on Ω is internal.

Now let $\mathcal{A} = {}^*\mathcal{P}(\Omega)$, and take μ to be the normalized counting measure on \mathcal{A} , defined by $\mu(A) = \frac{|A|}{|\Omega|}$ for all internal $A \subseteq \Omega$. We'll prove that the measure on G obtained from the Loeb counting measure $(\Omega, \mathcal{A}_L, \mu_L)$ by pushing down along the map $\text{st}: \Omega \rightarrow G$ is a left Haar measure on G . So we need to show:

- (1) If $V \subset G$ is open and nonempty, then (a) $\text{st}^{-1}(V) \in \mathcal{A}_L$ and (b) $\mu_L(\text{st}^{-1}(V)) > 0$.
- (2) $(\mu_L)_{\text{st}}$ is left invariant.

First we prove (1a); let V be open and consider the closed set $C = G \setminus V$. A straightforward argument shows

$$\text{st}^{-1}(C) = \bigcap \{ {}^*O \cap \Omega \mid O \text{ is } \tau\text{-open and } C \subseteq O \}.$$

Using Lemma 8.10 it follows that $\text{st}^{-1}(C) \in \mathcal{A}_L$, and hence $\text{st}^{-1}(V) \in \mathcal{A}_L$. Note that from (1a) we may conclude that $(\mathcal{A}_L)_{\text{st}}$ contains every Borel subset of G , since it is a σ -algebra.

Next, note that (1b) follows from (2) by compactness of G . Indeed, if V were a nonempty open set with measure 0, its translates comprise an open cover of G , so compactness would yield a finite cover of open sets of measure 0. This contradicts the fact that G has measure 1.

Next we prove (2); it suffices to prove (2') $(\mu_L)_{\text{st}}(gB) \geq (\mu_L)_{\text{st}}(B)$ for all Borel sets B and all $g \in G$. (We get the opposite inequality by replacing B by gB and g by g^{-1} .)

Given B and g , let $\epsilon > 0$ be standard and take an internal set $A \subseteq \text{st}^{-1}(B)$ such that ${}^\circ\mu(A) + \epsilon \geq \mu_L(\text{st}^{-1}(B))$. Define $S = \{h \in \Omega \mid hU \cap gaU \neq \emptyset \text{ for some } a \in A\}$. Note that S is internal.

First we prove $S \subseteq \text{st}^{-1}(gB)$. Take any $h \in S$ and select a and x such that $x \in hU \cap gaU$. We have $x = hb$ for some $b \in U \subseteq \mu(e)$, so $h^{-1}x \in \mu(e)$ and hence $h \approx x \approx ga \in g\text{st}^{-1}(B) = \text{st}^{-1}(gB)$, using the continuity of the group operations on G .

Next we prove $|S| \geq |A|$. Consider the internal family \mathcal{C} of $*$ open subsets of $*G$ given by

$$\mathcal{C} = \{hU \mid h \in \Omega \setminus A\} \cup \{g^{-1}hU \mid h \in S\}.$$

We'll show this covers $*G$. If so, then by the choice of Ω we'd have $|\Omega \setminus A| + |A| = |\Omega| \leq |\Omega \setminus A| + |S|$ and hence $|A| \leq |S|$. So it remains to show \mathcal{C} covers $*G$. We know that $\{hU \mid h \in \Omega\}$ covers $*G$. Thus it suffices to show for each $a \in A$ that the set aU is covered by $\{g^{-1}hU \mid h \in S\}$. To that end, suppose $a \in A$ and $x \in aU$; then $gx \in hU$ for some $h \in \Omega$, and for that h we have $gx \in hU \cap gaU$ and hence $h \in S$. Then note $x = g^{-1}gx \in g^{-1}hU \subseteq \cup\{g^{-1}hU \mid h \in S\}$, as desired.

This completes the proof that \mathcal{C} covers $*G$, and hence proves that $|S| \geq |A|$.

To complete the proof of (2'), note that

$$(\mu_L)_{\text{st}}(gB) \geq {}^\circ\mu(S) \geq {}^\circ\mu(A) \geq (\mu_L)_{\text{st}}(B) - \epsilon.$$

Letting ϵ tend to 0 yields (2'). □

This construction of the Haar measure yields some further properties easily:

8.21. Proposition. *The left Haar measure on G constructed in the previous result is inner regular. That is, for any $S \subseteq G$ such that $S \in (\mathcal{A}_L)_{\text{st}}$ and any $\epsilon > 0$, there exists a closed $C \subseteq S$ such that $(\mu_L)_{\text{st}}(S \setminus C) < \epsilon$.*

Proof. Since $\text{st}^{-1}(S) \in \mathcal{A}_L$, there is an internal set $A \subseteq \text{st}^{-1}(S)$ with $\mu_L(\text{st}^{-1}(S) \setminus A) < \epsilon$. Let C be the closed set $\text{st}(A)$. Clearly $C \subseteq S$, and $\text{st}^{-1}(S \setminus C) \subseteq \text{st}^{-1}(S) \setminus A$. Hence $(\mu_L)_{\text{st}}(S \setminus C) = \mu_L(\text{st}^{-1}(S \setminus C)) \leq \mu_L(\text{st}^{-1}(S) \setminus A) < \epsilon$. \square