## Liftings

In this subsection we consider an internal measure space $(\Omega, \mathcal{A}, \mu)$ with $\mu(\Omega)$ finite. In order to work effectively with the Loeb measure $\mu_L$, we need to understand the $\mathcal{A}_L$-measurable functions $f\colon \Omega \to \mathbb{R}$. It turns out that they can be represented by well behaved internal functions $F\colon \Omega \to {}^*\mathbb{R}$, as is made precise in the next definition and theorem. This provides the basis for our development of integration theory for the Loeb measure space $(\Omega, \mathcal{A}_L, \mu_L)$, which we do in the next section.

It is useful to consider extended real-valued functions $f\colon \Omega \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}}$ denotes the set $\mathbb{R} \cup \{-\infty, +\infty\}$. As usual we regard $\bar{\mathbb{R}}$ as a linearly ordered set extending $(\mathbb{R}, <)$ in which $+\infty$ is the maximum and $-\infty$ is the minimum element. Furthermore, we formally extend the standard part map to all of ${}^*\mathbb{R}$ by taking $\mathrm{st}(u) = +\infty$ when $u \in {}^*\mathbb{R}$ is positive infinite and $\mathrm{st}(u) = -\infty$ when it is negative infinite.

8.22. **Definition.** Let $f\colon \Omega \to \bar{\mathbb{R}}$ and $F\colon \Omega \to {}^*\mathbb{R}$ be functions. We say $F$ *is a lifting of* $f$ with respect to the internal measure space $(\Omega, \mathcal{A}, \mu)$ if the following conditions hold:
(1) $F$ is internal and its range is a hyperfinite set;
(2) for each $u$ in the range of $F$, the set $F^{-1}(u)$ is in $\mathcal{A}$;
(3) $f(x) = \mathrm{st}(F(x))$ for $\mu_L$-almost every $x \in \Omega$.

We prove that the functions with liftings are exactly the $\mathcal{A}_L$-measurable functions. To do that we first need to prove the following technical lemma.

8.23. **Lemma.** *Suppose $S, S_1, S_2$ are sets in $\mathcal{A}_L$ with $S_1 \subseteq S \subseteq S_2$. Suppose further that $A_1, A_2 \in \mathcal{A}$ are such that $A_1 \subseteq A_2$ and $S_i \triangle A_i$ is a $\mu_L$-null set for $i = 1, 2$. Then there exists $A \in \mathcal{A}$ such that $A_1 \subseteq A \subseteq A_2$ and $S \triangle A$ is a $\mu_L$-null set*

*Proof.* Since $S \in \mathcal{A}_L$, there exists $B \in \mathcal{A}$ such that $S \triangle B$ is a $\mu_L$-null set. Let $A = (B \cap A_2) \cup A_1$. Clearly $A \in \mathcal{A}$ and $A_1 \subseteq A \subseteq A_2$. Noting that $S = (S \cap S_2) \cup S_1$, a simple calculation using properties of the symmetric difference operation shows

$$S \triangle A \subseteq (S \triangle B) \cup (S_1 \triangle A_1) \cup (S_2 \triangle A_2)$$

and thus $S \triangle A$ is a $\mu_L$-null set, as desired. □

8.24. **Theorem.** *A function* $f\colon \Omega \to \bar{\mathbb{R}}$ *has a lifting with respect to the internal measure space* $(\Omega, \mathcal{A}, \mu)$ *if and only if* $f$ *is* $\mathcal{A}_L$*-measurable.*

*Proof.* ($\Rightarrow$) Suppose $f\colon \Omega \to \bar{\mathbb{R}}$ has the function $F\colon \Omega \to {}^*\mathbb{R}$ as a lifting with respect to $(\Omega, \mathcal{A}, \mu)$. Let $N = \{x \in \Omega \mid f(x) \neq \mathrm{st}(F(x))\}$, so $N$ is a $\mu_L$-null set. Fix $r \in \mathbb{R}$; we must show that the set $\{x \in \Omega \mid f(x) \leq r\}$ is in $\mathcal{A}_L$. For $x$ outside $N$ we know $f(x) = \mathrm{st}(F(x))$ and therefore we have

$$f(x) \leq r \quad \Leftrightarrow \quad F(x) \leq r + \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Furthermore, the assumption that $F$ is a lifting and the fact that $\mathcal{A}$ is closed under hyperfinite boolean operations implies that for each $n \in \mathbb{N}$

$$\{x \in \Omega \mid F(x) \leq r + \frac{1}{n}\} \in \mathcal{A}.$$

Hence

$$\bigcap_{n \in \mathbb{N}} \{x \in \Omega \mid F(x) \leq r + \frac{1}{n}\} \in \mathcal{A}_L$$

and the symmetric difference between this set and $\{x \in \Omega \mid f(x) \leq r\}$ is a subset of $N$, so it is a $\mu_L$-null set. This implies that $\{x \in \Omega \mid f(x) \leq r\} \in \mathcal{A}_L$ for every $r \in \mathbb{R}$; it follows that $f$ is $\mathcal{A}_L$-measurable.

($\Leftarrow$) Let $f\colon \Omega \to \bar{\mathbb{R}}$ be $\mathcal{A}_L$-measurable. Let $\bar{\mathbb{Q}} = \mathbb{Q} \cup \{-\infty, +\infty\}$. For each $q \in \bar{\mathbb{Q}}$ let $S_q = \{x \in \Omega \mid f(x) \leq q\}$. Note that these sets have the following properties: $S_q \in \mathcal{A}_L$ for all $q \in \bar{\mathbb{Q}}$; $S_q \subseteq S_r$ for all $q \leq r$ in $\bar{\mathbb{Q}}$; and $\cup(S_q \mid q \in \bar{\mathbb{Q}}) = S_{+\infty} = \Omega$.

We may construct a family $(A_q \mid q \in \bar{\mathbb{Q}})$ from $\mathcal{A}$ with the following properties: $S_q \triangle A_q$ is a $\mu_L$-null set for all $q \in \bar{\mathbb{Q}}$; $A_q \subseteq A_r$ for all $q \leq r$ in $\bar{\mathbb{Q}}$; and $\cup(A_q \mid q \in \bar{\mathbb{Q}}) = A_{+\infty} = \Omega$. To do this we first set $A_{+\infty} = \Omega$ and choose $A_{-\infty} \in \mathcal{A}$ so that $S_{-\infty} \triangle A_{-\infty}$ is a $\mu_L$-null set, and then we proceed to define $A_q$ for $q \in \mathbb{Q}$ by induction over an $\omega$-list of the elements of $\mathbb{Q}$, using the preceding lemma at each step.

By $\omega_1$-saturation there is a hyperfinite set $P \subseteq {}^*\bar{\mathbb{Q}}$ and an internal function $G\colon P \to \mathcal{A}$ with the following properties: $\bar{\mathbb{Q}} \subseteq P$ and $G(q) = A_q$ for all $q \in \bar{\mathbb{Q}}$; for all $q, r \in P$ with $q \leq r$ we have $G(q) \subseteq G(r)$. When $q \in P$ we will write $A_q$ for $G(q)$.

Let $P' = P \setminus \{-\infty, +\infty\} = P \cap {}^*\mathbb{Q}$, so $P'$ is hyperfinite and $\mathbb{Q} \subseteq P' \subseteq {}^*\mathbb{Q}$. Choose $q_{-\infty}, q_{+\infty} \in {}^*\mathbb{Q}$ such that $q_{-\infty} < P' < q_{+\infty}$.

Now we are ready to define the function $F\colon \Omega \to {}^*\mathbb{R}$ that will be a lifting of the given $f$:

$$F(x) = \begin{cases} q_{+\infty} & \text{if } x \notin A_q \text{ for all } q \in P' \\ \min\{q \in P' \mid x \in A_q\} & \text{if } x \in A_q \text{ for some } q \in P' \\ & \qquad\qquad \text{but } x \notin A_{-\infty} \\ q_{-\infty} & \text{if } x \in A_{-\infty} \end{cases}$$

The Internal Definition Principle yields that $F$ is internal, and the range of $F$ is a subset of $P' \cup \{q_{-\infty}, q_{+\infty}\}$, so it is hyperfinite. Each set $F^{-1}(x)$ is a boolean combination of the sets $(A_q \mid q \in P)$, so it is an element of $\mathcal{A}$.

It remains only to show that $\mathrm{st}(F(x)) = f(x)$ holds for $\mu_L$-almost every $x \in \Omega$. Consider the set

$$N = \bigcup_{q \in \mathbb{Q}} S_q \triangle A_q.$$

Since this is a countable union of $\mu_L$-null sets, we have that $N$ is also a $\mu_L$-null set. Note that for any $x \in \Omega \setminus N$ and any $q \in \mathbb{Q}$ (so $q$ is standard!) we have

$$f(x) \leq q \iff x \in S_q \iff x \in A_q \iff F(x) \leq q.$$

From this we can deduce that $f(x) = \mathrm{st}(F(x)$ for all $x \in \Omega \setminus N$. Indeed, for such $x$ we have the following equivalences:

$$\begin{aligned} f(x) = +\infty &\iff f(x) \not\leq q \ \text{ holds, for all } q \in \mathbb{Q} \\ &\iff F(x) \not\leq q \ \text{ holds, for all } q \in \mathbb{Q} \\ &\iff \mathrm{st}(F(x)) = +\infty. \end{aligned}$$

$$\begin{aligned} f(x) = -\infty &\iff f(x) \leq q \ \text{ holds, for all } q \in \mathbb{Q} \\ &\iff F(x) \leq q \ \text{ holds, for all } q \in \mathbb{Q} \\ &\iff \mathrm{st}(F(x)) = -\infty. \end{aligned}$$

Finally, for each $r \in \mathbb{R}$

$$\begin{aligned} f(x) \leq r &\iff f(x) \leq q \ \text{ holds, for all } q \in \mathbb{Q} \cap (r, \infty) \\ &\iff F(x) \leq q \ \text{ holds, for all } q \in \mathbb{Q} \cap (r, \infty) \\ &\iff \mathrm{st}(F(x)) \leq r. \end{aligned}$$

This obviously implies $f(x) = \mathrm{st}(F(x))$ whenever $f(x) \in \mathbb{R}$. □

8.25. **Remark.** The proof of ($\Rightarrow$) in the previous result can be used to show that if $F\colon \Omega \to {}^*\mathbb{R}$ is internal and $\mathcal{A}$-measurable in a certain sense, and if $f\colon \Omega \to \bar{\mathbb{R}}$ satisfies $f(x) = \mathrm{st}(F(x))$ for $\mu_L$-almost every $x \in \Omega$, then $f$ is $\mathcal{A}_L$-measurable. (The measurability of $F$ that we require is that for every standard $r \in \mathbb{R}$, the set $F^{-1}({}^*(-\infty, r])$ belongs to $\mathcal{A}_L$. In many situations this internal set will in fact belong to $\mathcal{A}$.)

The family of liftings has many useful closure properties. In the next result we give a few that will be useful in the next section. In the statement of this result, we give $\bar{\mathbb{R}}$ the order topology. This makes it a compact Hausdorff space homeomorphic to a closed bounded interval in $\mathbb{R}$. The standard part map of $\bar{\mathbb{R}}$ for this topology is obtained from our extended map $\mathrm{st}\colon {}^*\mathbb{R} \to \bar{\mathbb{R}}$ by simply setting $\mathrm{st}(+\infty) = +\infty$ and $\mathrm{st}(-\infty) = -\infty$. (Note that ${}^*\bar{\mathbb{R}} = {}^*\mathbb{R} \cup \{-\infty, +\infty\}$.)

8.26. **Proposition.** *Suppose $f\colon \Omega \to \bar{\mathbb{R}}$ has $F\colon \Omega \to {}^*\mathbb{R}$ as a lifting. If $\alpha\colon \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is continuous and $\alpha(\mathbb{R}) \subseteq \mathbb{R}$, then ${}^*\alpha \circ F$ is a lifting of $\alpha \circ f$. In particular, for any $r \in \mathbb{R}$*
*(1)* $\min(r, F)$ *is a lifting of* $\min(r, f)$*;*
*(2)* $\max(r, F)$ *is a lifting of* $\max(r, f)$*;*
*(3)* $rF$ *is a lifting of* $rf$*;*
*(4)* $|F|$ *is a lifting of* $|f|$.

*Proof.* Suppose $f, F, \alpha$ have the stated properties. It is obvious that ${}^*\alpha \circ F$ is internal and has hyperfinite range contained in ${}^*\mathbb{R}$. Given any $u$ in the range of ${}^*\alpha \circ F$, we note that

$$({}^*\alpha \circ F)^{-1}(u) = F^{-1}({}^*\alpha^{-1}(u) \cap \text{range of } F)$$

and since $\mathcal{A}$ is closed under hyperfinite unions, $({}^*\alpha \circ F)^{-1}(u) \in \mathcal{A}$. Finally, the fact that $\alpha$ is continuous ensures that

$$\mathrm{st}({}^*\alpha \circ F(x)) = \alpha(\mathrm{st}(F(x)))$$

for any $x \in \Omega$. Thus $\mathrm{st}({}^*\alpha \circ F(x)) = \alpha(f(x))$ for any $x \in \Omega$ for which $\mathrm{st}(F(x)) = f(x)$. It follows that ${}^*\alpha \circ F$ is a lifting of $\alpha \circ f$.

The rest of the Proposition follows from the fact that the functions given respectively by $\alpha(y) = \min(r, y), \max(r, y), ry, \mathrm{or}|y|$ satisfy the hypotheses. (Here we define $|-\infty| = |+\infty| = +\infty$.) ☐

8.27. **Remark.** Suppose $a, b \in \mathbb{R}$ satisfy $a \leq b$. The previous result can be used to show that if $F\colon \Omega \to {}^*[a, b]$ is a lifting of $f\colon \Omega \to [a, b]$ and if $\alpha\colon [a, b] \to \mathbb{R}$ is continuous, then ${}^* \circ F$ is a lifting of $\alpha \circ f$. (Just extend $\alpha$ to a continuous function from $\bar{\mathbb{R}}$ to $\bar{\mathbb{R}}$ by taking it to be constant on $[-\infty, a]$ and on $[b, \infty]$ and apply the previous result.)