9. Integration

In this section we develop integration theory for finite Loeb measures. We work in a nonstandard extension $^*\colon U(X) \to U(^*X)$ which is assumed to be at least ω_1-saturated.

Fix an internal measure space $(\Omega, \mathcal{A}, \mu)$ in $U(^*X)$ such that $\mu(\Omega)$ is finite.

9.1. **Definition.** An $\mathcal{A}$-*simple function* is an internal function $F\colon \Omega \to {}^*\mathbb{R}$ whose range is a hyperfinite set S such that for each u in S, the set $F^{-1}(\{u\})$ is in $\mathcal{A}$.

9.2. **Remark.** By definition, each lifting of a function from Ω into $\bar{\mathbb{R}}$ is an $\mathcal{A}$-simple function. Further, if $F\colon \Omega \to {}^*\mathbb{R}$ is an $\mathcal{A}$-simple function, then F is a lifting of the function $\operatorname{st} F\colon \Omega \to \bar{\mathbb{R}}$ and $\operatorname{st} F$ is $\mathcal{A}_L$-measurable. (See the first part of the proof of Theorem 8.24.)

We begin by defining an internal integral for $\mathcal{A}$-simple functions. Let $F\colon \Omega \to {}^*\mathbb{R}$ be $\mathcal{A}$-simple and let S be the range of F. Since S is hyperfinite and the function on S mapping u to $u \cdot \mu(F^{-1}(\{u\}))$ is internal, we may use the hyperfinite summation operation to define:

$$\int F\, d\mu \quad = \quad \sum_{u \in S} u \cdot \mu(F^{-1}(\{u\})).$$

For any set $A \in \mathcal{A}$ the function $F \cdot \chi_A$ is also $\mathcal{A}$-simple, and we define

$$\int_A F\, d\mu \quad = \quad \int F \cdot \chi_A\, d\mu.$$

(Here χ_A is the characteristic function of A; thus $F \cdot \chi_A$ agrees with F on A and is 0 on $\Omega \setminus A$.)

9.3. **Proposition** (Properties of the internal integral)**.** *Let F, G be $\mathcal{A}$-simple functions and $c \in {}^*\mathbb{R}$. Then*
(1) $\int c \cdot F\, d\mu \ = \ c \int F\, d\mu$;
(2) $\int (F+G)\, d\mu \ = \ \int F\, d\mu + \int G\, d\mu$;
(3a) if $F(x) \le G(x)$ for all $x \in \Omega$, then $\int F\, d\mu \le \int G\, d\mu$;
(3b) in particular, $\left|\int F\, d\mu\right| \le \int |F|\, d\mu$;
(4) if $F(x) \ge 0$ for all $x \in \Omega$, then the map $A \mapsto \int_A F\, d\mu$ is an internal finitely additive measure on $\mathcal{A}$.

Proof. These proofs mainly depend on simple properties of the hyperfinite summation operation, which can easily be verified by transfer. □

The next Theorem is the main result in this section; it provides the foundation of integration theory for finite Loeb measures.

9.4. **Theorem.** *Let* $F\colon \Omega \to {}^*\mathbb{R}$ *be an* $\mathcal{A}$*-simple function. The following conditions are equivalent:*
(1) $\int |F|\,d\mu$ *is finite and*

$$\operatorname{st}\Big(\int |F|\,d\mu\Big) = \lim_{n\to\infty} \operatorname{st}\Big(\int \min(|F|, n)\,d\mu\Big);$$

(2) For each standard $\epsilon > 0$ *there exists a standardly bounded* $\mathcal{A}$*-simple function* G *such that*

$$\int |F - G|\,d\mu \;<\; \epsilon;$$

(3) $\int |F|\,d\mu$ *is finite and for all* $A \in \mathcal{A}$*,*

$$\text{if } \mu(A) \approx 0, \text{ then } \int_A |F|\,d\mu \approx 0;$$

(4) if $u \in {}^*\mathbb{R}$ *is positive infinite, and we set* $A = \{x \in \Omega \mid |F(x)| > u\}$*, then*

$$\int_A |F|\,d\mu \approx 0.$$

Proof. ($1 \Rightarrow 2$) Fix a standard $\epsilon > 0$ and choose $n \in \mathbb{N}$ large enough so that the internal integrals of $|F|$ and $\min(|F|, n)$ differ by less than ϵ, which is possible by (1). Define $G\colon \Omega \to {}^*\mathbb{R}$ by letting $G(x) = F(x)$ when $-n \le F(x) \le n$ and by letting $G(x) = n$ if $F(x) > n$ and $G(x) = -n$ if $F(x) < n$. Evidently G is standardly bounded and $\mathcal{A}$-simple. One checks easily that $|G| = \min(|F|, n) \le |F|$ and that $|F - G| = |F| - |G|$. Therefore

$$\int |F - G|\,d\mu = \int |F|\,d\mu - \int |G|\,d\mu < \epsilon$$

as desired.

($2 \Rightarrow 3$) Fix a standard $\epsilon > 0$ and use (2) to get a standardly bounded $\mathcal{A}$-simple function G such that the internal integral of $|F - G|$ is less than ϵ. Let $K \in \mathbb{R}$ satisfy $|G(x)| \le K$ for all $x \in \Omega$. For any $A \in \mathcal{A}$ we have

$$\begin{aligned} 0 \le \int_A |F|\,d\mu &\le \int_A |F - G|\,d\mu + \int_A |G|\,d\mu \\ &\le \int |F - G|\,d\mu + \int_A K\,d\mu \le \epsilon + K \cdot \mu(A). \end{aligned}$$

Taking $A = \Omega$ and any ϵ we see this forces the internal integral of $|F|$ to be finite. For $A \in \mathcal{A}$ with $\mu(A) \approx 0$ we see that

$$0 \leq \int_A |F|\, d\mu \leq \epsilon + K \cdot \mu(A) < 2\epsilon.$$

Since ϵ was an arbitrary standard positive number, this yields that $\int_A |F|\, d\mu$ must be infinitesimal.

($3 \Rightarrow 4$) Let $u \in {}^*\mathbb{R}$ be positive infinite and set $A = \{x \in \Omega \mid |F(x)| > u\}$. Note that $A \in \mathcal{A}$. It suffices to show $\mu(A) \approx 0$, since we can then apply (3) to conclude $\int_A |F|\, d\mu \approx 0$. We have

$$u \cdot \mu(A) \leq \int_A |F|\, d\mu \leq \int |F|\, d\mu$$

and thus

$$\mu(A) \leq \left(\frac{1}{u}\right) \cdot \int |F|\, d\mu \approx 0$$

so $\mu(A) \approx 0$ as desired.

($4 \Rightarrow 1$) First we show the internal integral of F is finite. Otherwise there exists a positive infinite $u \in {}^*\mathbb{R}$ with $2u < \int |F|\, d\mu$. Choose an infinite $N \in {}^*\mathbb{N}$ such that $N \cdot \mu(\Omega) \leq u$, which is possible since $\mu(\Omega)$ is assumed to be finite. Let $A = \{x \in \Omega \mid |F(x)| > N\} \in \mathcal{A}$. Then we have

$$2u < \int |F|\, d\mu = \int_A |F|\, d\mu + \int_{A^c} |F|\, d\mu \leq \int_A |F|\, d\mu + N \cdot \mu(\Omega) < 1 + u$$

where, in the last inequality we used (4). This yields $u < 1$, which is a contradiction.

Therefore, if (4) holds but (1) fails, there must exist standard $r < s$ in $\mathbb{R}$ such that for all standard $n \in \mathbb{N}$ we have

$$\int \min(|F|, n)\, d\mu < r < s < \int |F|\, d\mu.$$

Since this condition on n is internal, by overspill there exists an infinite $N \in {}^*\mathbb{N}$ such that

$$\int \min(|F|, N)\, d\mu < r.$$

Again let $A = \{x \in \Omega \mid |F(x)| > N\} \in \mathcal{A}$. Note that using (4) we have

$$\begin{aligned} \int |F|\, d\mu &= \int_A |F|\, d\mu + \int_{A^c} |F|\, d\mu \approx \int_{A^c} |F|\, d\mu \\ &\leq \int \min(|F|, N)\, d\mu < r < s < \int |F|\, d\mu. \end{aligned}$$

Taking standard parts yields a contradiction. □

9.5. **Definition.** We say that an $\mathcal{A}$-simple function F is *S-integrable* if it satisfies the four equivalent conditions in Theorem 9.4.

9.6. **Proposition** (Properties of S-integrable functions)**.** *Let F, G be S-integrable functions and let $c \in {}^*\mathbb{R}$ be finite. Then:*
(1) Every standardly bounded $\mathcal{A}$-simple function is S-integrable. In particular, the constant function with value c is S-integrable.
(2) If H is an $\mathcal{A}$-simple function and $|H(x)| \leq |F(x)|$ for all $x \in \Omega$, then H is S-integrable.
(3) The functions $|F|, cF, \max(F, G), \min(F, G)$ and $F+G$ are S-integrable. In particular, the positive and negative parts of F (which are defined by $F^+ = \max(0, F)$ and $F^- = -\min(F, 0)$) are S-integrable.

Proof. In this proof we refer several times to conditions (1)–(4) from Theorem 9.4.

(1) If H is a standardly bounded $\mathcal{A}$-simple function and $u \in {}^*\mathbb{R}$ is positive infinite, then $\{x \in \Omega \mid |H(x)| > u\} = \emptyset$ and therefore H trivially satisfies condition (4).

(2) If F, H are as assumed in (2) and $A \in \mathcal{A}$, then

$$0 \leq \int_A |H| \, d\mu \ \leq \ \int_A |F| \, d\mu.$$

This makes it clear that condition (4) for F implies (4) for H.

(3) The S-integrability of $|F|$ is an immediate consequence of how conditions (1),(3) and (4) are stated. The S-integrability of cF can be checked easily using condition (3) or (4) since c is finite and the internal integral is linear. To treat $H = \max(F, G)$, note that there exists $B \in \mathcal{A}$ such that $H = F$ on B and $H = G$ on B^c. For any $A \in \mathcal{A}$ we therefore have

$$\int_A |H| \, d\mu = \int_{A \cap B} |F| \, d\mu + \int_{A \cap B^c} |G| \, d\mu \leq \int_A |F| \, d\mu + \int_A |G| \, d\mu.$$

This makes it easy to check condition (3) for H. A similar argument handles $\min(F, G)$. Treating $F + G$ is even easier, using the triangle inequality $|F + G| \leq |F| + |G|$. □

9.7. **Proposition.** *Let F, G be S-integrable functions. If the set $\{x \in \Omega \mid F(x) \not\approx G(x)\}$ has μ_L-measure 0, then*

$$\int F \, d\mu \ \approx \ \int G \, d\mu.$$

Proof. If the hypotheses hold, then for each $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}$ such that $\mu(A_n) < \frac{1}{n}$ and $\{x \in \Omega \mid F(x) \not\approx G(x)\} \subseteq A_n$. In particular, we have that $|F(x) - G(x)| < \frac{1}{n}$ for all $x \in \Omega \setminus A_n$. Using ω_1-saturation we get a set $A \in \mathcal{A}$ and a positive infinitesimal $\delta \in {}^*\mathbb{R}$ such that $\mu(A) < \delta$ and $|F(x) - G(x)| < \delta$ for all $x \in \Omega \setminus A$. Then we have

$$\int |F-G|\,d\mu = \int_A |F-G|\,d\mu + \int_{A^c} |F-G|\,d\mu \leq \int_A |F-G|\,d\mu + \delta \cdot \mu(\Omega).$$

The right side of the last inequality is a sum of two infinitesimals (applying condition (3) of Theorem 9.4 to the S-integrable function $F - G$ and using the assumption that $\mu(\Omega)$ is finite) and therefore we have $\int |F-G|\,d\mu \approx 0$. The conclusion of the Proposition follows immediately. □

9.8. **Remark.** The assumption of S-integrability is essential for the conclusion of the previous proposition to be true. For example, suppose there is a set $A \in \mathcal{A}$ for which $\mu(A)$ is a positive infinitesimal. Let u be a positive number in ${}^*\mathbb{R}$ and let F_u be the function that is equal to u on A and equal to 0 on $\Omega \setminus A$. Note that $\int F_u\,d\mu = u \cdot \mu(A)$, which can be made equal to any positive number in ${}^*\mathbb{R}$ by choosing u suitably. If $u \cdot \mu(A)$ is not infinitesimal, then $\{x \in \Omega \mid F_u(x) \not\approx 0\} = A$ is a μ_L-null set but $\int F_u\,d\mu \not\approx \int 0\,d\mu$.

Based on the previous Proposition, we may define an integral with respect to the Loeb measure μ_L for any function $f \colon \Omega \to \bar{\mathbb{R}}$ that has an S-integrable lifting:

9.9. **Definition.** Suppose $f \colon \Omega \to \bar{\mathbb{R}}$ has an S-integrable lifting. We define

$$\int f\,d\mu_L = \mathrm{st}\,\Big(\int F\,d\mu\Big)$$

where F is any S-integrable lifting of f.