Next we want to compare the μ_L-integral defined above to the usual integral defined for the measure space $(\Omega, \mathcal{A}_L, \mu_L)$. For any $\mathcal{A}_L$-measurable function $f\colon \Omega \to \bar{\mathbb{R}}$, let $\Phi(f)$ denote the integral of f as defined in the usual developments of measure theory. Let $L_1(\Omega, \mathcal{A}_L, \mu_L)$ denote the usual space of integrable functions, which consists of the $\mathcal{A}_L$-measurable functions f for which the integral $\Phi(|f|)$ is finite. (See below for a discussion of how $\Phi(f)$ is defined.) We show that these two integration theories are identical. That is, for an $\mathcal{A}_L$-measurable function $f\colon \Omega \to \bar{\mathbb{R}}$, we show that f is in $L_1(\Omega, \mathcal{A}_L, \mu_L)$ if and only if f has an S-integrable lifting and, moreover, if F is an S-integrable lifting of f, then $\Phi(f) = \operatorname{st}\big(\int F\, d\mu\big)$.

One key idea for proving this equivalence is contained in the following simple approximation result, which gives another characterization of S-integrable functions:

9.10. **Proposition.** *Let $F\colon \Omega \to {}^*\mathbb{R}$ be an $\mathcal{A}$-simple function.*

Then F is S-integrable if and only if for each standard $\epsilon > 0$ there exists an $\mathcal{A}$-simple function G such that $|G(x)| \le |F(x)|$ for all $x \in \Omega$, the range of G is a finite set of standard real numbers, and $\int |F - G|\, d\mu < \epsilon$.

Proof. $(\Leftarrow)$ An $\mathcal{A}$-simple function F with this property obviously satisfies (2) in Theorem 9.4. In particular, note that for any G with the stated properties, the integral $\int |G|\, d\mu$ is finite, which forces $\int |F|\, d\mu$ to also be finite.

$(\Rightarrow)$ By treating the positive and negative parts of F separately, we reduce to the case where F also satisfies $F(x) \ge 0$ for all $x \in \Omega$.

Fix a standard $\epsilon > 0$. Since F is S-integrable, there exists $n \in \mathbb{N}$ such that the integrals of F and of $\min(F, n)$ differ by at most $\epsilon/2$. Choose $k \in \mathbb{N}$ larger than $2\mu(\Omega)/\epsilon$. Let $g\colon [0, n] \to \{0, 1/k, 2/k, \dots, (nk)/k\}$ be the function that takes x to the largest fraction of the form m/k that is $\le x$. Finally let G be the composition ${}^*g \circ \min(F, n)$. It is easy to check that G is an $\mathcal{A}$-simple function whose values lie in the set $\{0, 1/k, 2/k, \dots, (nk)/k\}$ and that $0 \le G(x) \le \min(F(x), n) \le F(x)$ and $|G(x) - \min(F(x), n)| \le 1/k$ hold for all $x \in \Omega$. The last inequality together with the choice of k imply that $\int |G - \min(F, n)|\, d\mu < \epsilon/2$ and therefore $\int |F - G|\, d\mu < \epsilon$. Therefore G has all of the desired properties. □

9.11. **Corollary.** *If* $f\colon \Omega \to \bar{\mathbb{R}}$ *has an* S*-integrable lifting, then* $f \in L_1(\Omega, \mathcal{A}_L, \mu_L)$*. Moreover, if* F *is an* S*-integrable lifting of* f*, then*

$$\Phi(f) = \operatorname{st}\Big(\int F\, d\mu\Big).$$

Proof. In proving this result we use the following definition of $\Phi(f)$ (see Chapter 2 of the book *Real Analysis* by G. Folland, for example): first, for an $\mathcal{A}_L$-simple function g, $\Phi(g)$ is defined in the obvious way:

$$\Phi\Big(\sum_{j=1}^{n} r_j \cdot \chi_{X_j}\Big) = \sum_{j=1}^{n} r_j \cdot \mu_L(X_j)$$

for any pairwise disjoint sets $X_1, \dots, X_n$ from $\mathcal{A}_L$ and real numbers $r_1, \dots, r_n$. Second, for an arbitrary $\mathcal{A}_L$-measurable $f\colon \Omega \to [0, \infty]$, the integral $\Phi(f)$ is defined to be the supremum of the values $\Phi(g)$ where g is any $\mathcal{A}_L$-simple function such that $0 \leq g(x) \leq f(x)$ for all $x \in \Omega$. Such a function f is taken to be an element of $L_1(\Omega, \mathcal{A}_L, \mu_L)$ iff $\Phi(f)$ is finite. For an arbitrary $\mathcal{A}_L$-measurable function f we put f in $L_1(\Omega, \mathcal{A}_L, \mu_L)$ iff the nonnegative functions $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are in $L_1(\Omega, \mathcal{A}_L, \mu_L)$, and in that case we set $\Phi(f) = \Phi(f^+) - \Phi(f^-)$.

Now take f to be any function with an S-integrable lifting F. By separating f into its positive and negative parts, we reduce to the case where $f(x) \geq 0$ for all $x \in \Omega$. We may likewise assume that F satisfies $F(x) \geq 0$ for all $x \in \Omega$.

Fix a standard $\epsilon > 0$. Applying the previous proposition to F we obtain $G\colon \Omega \to {}^*\mathbb{R}$ that is $\mathcal{A}$-simple, has range a finite set of standard real numbers, satisfies $0 \leq G(x) \leq F(x)$ for all $x \in \Omega$, and has $\int |F - G|\, d\mu < \epsilon$. Let $N = \{x \in \Omega \mid F(x) \not\approx f(x)\}$, so N is a μ_L-null set. Finally, let $G'\colon \Omega \to \mathbb{R}$ be equal to 0 on N and equal to G on $\Omega \setminus N$. Then G' is an $\mathcal{A}_L$-simple function and satisfies $0 \leq G'(x) \leq f(x)$ for all $x \in \Omega$; moreover, $\Phi(G') = \Phi(G) \approx \int G\, d\mu$. Also we have $0 \leq \int G\, d\mu \leq \int F\, d\mu < \int G\, d\mu + \epsilon$, from which it follows that

$$\operatorname{st}\Big(\int F\, d\mu\Big) \leq \Phi(G') + \epsilon \leq \Phi(f) + \epsilon.$$

Since ϵ was arbitrary we conclude $\operatorname{st}\big(\int F\, d\mu\big) \leq \Phi(f)$.

To complete the proof we assume $\operatorname{st}\big(\int F\, d\mu\big) < \Phi(f)$ and derive a contradiction. Take a standard $\epsilon > 0$ such that $\operatorname{st}\big(\int F\, d\mu\big) + 3\epsilon < \Phi(f)$. There exists an $\mathcal{A}_L$-simple function

$$g = \sum_{j=1}^{n} r_j \cdot \chi_{X_j}$$

with $X_1, \ldots, X_n$ pairwise disjoint sets in $\mathcal{A}_L$ and $r_1, \ldots, r_n > 0$ in $\mathbb{R}$, such that $0 \leq g(x) \leq f(x)$ for all $x \in \Omega$ and

$$\mathrm{st}\Big(\int F\,d\mu\Big) + 2\epsilon < \Phi(g) \leq \Phi(f).$$

Choose a standard $\delta > 0$ satisfying $2\delta \cdot \max(r_1, \ldots, r_n) < \epsilon$. For each $j = 1, \ldots, n$ choose $A_j \in \mathcal{A}$ such that $A_j \subseteq X_j$ and $\mu_L(X_j) - {}^\circ\mu(A_j) < \delta$. Further, choose $B \in \mathcal{A}$ such that ${}^\circ\mu(B) < \delta$ and $f(x) \approx F(x)$ for all $x \in \Omega \backslash B$; this is possible because F is a lifting of f. Finally set $B_j = A_j \backslash B$ for all $j = 1, \ldots, n$. Then $B_1, \ldots, B_n$ are pairwise disjoint, and $B_j \subseteq X_j$ and $\mu_L(X_j) - {}^\circ\mu(B_j) < 2\delta$ for $j = 1, \ldots, n$. We set

$$g' = \sum_{j=1}^{n} r_j \cdot \chi_{B_j}.$$

For $x \in B_1 \cup \cdots \cup B_n$ we have

$$g'(x) = g(x) \leq f(x) = \mathrm{st}(F(x))$$

and for $x \in \Omega$ outside $B_1 \cup \cdots \cup B_n$ we have $g'(x) = 0 \leq g(x)$. In particular, there exists an infinitesimal $\eta \geq 0$ such that $g'(x) \leq F(x) + \eta$ for all $x \in \Omega$ and therefore

$$\Phi(g') = \mathrm{st}\Big(\int g'\,d\mu\Big) \leq \mathrm{st}\Big(\int F\,d\mu\Big).$$

Note that

$$\Phi(g) - \Phi(g') = \sum_{j=1}^{n} r_j \cdot \mu_L(X_j \setminus B_j) \leq \max(r_1, \ldots, r_n) \cdot 2\delta < \epsilon$$

and therefore

$$\mathrm{st}\Big(\int F\,d\mu\Big) + \epsilon < \Phi(g').$$

This contradiction completes the proof. □

Finally, with a little more argument we are ready to prove the main result in this subsection:

9.12. **Theorem.** *Let $f\colon \Omega \to \bar{\mathbb{R}}$ be $\mathcal{A}_L$-measurable. Then f is in $L_1(\Omega, \mathcal{A}_L, \mu_L)$ if and only if f has an S-integrable lifting. Moreover, if F is an S-integrable lifting of f, then $\Phi(f) = \mathrm{st}\big(\int F\,d\mu\big)$.*

Proof. Given the previous corollary, it remains only to show that every f in $L_1(\Omega, \mathcal{A}_L, \mu_L)$ has an S-integrable lifting. As in the previous arguments, we may assume $f(x) \geq 0$ for all $x \in \Omega$. Because f is $\mathcal{A}_L$-measurable, it has a lifting F, which we may assume also satisfies $F(x) \geq 0$ for all $x \in \Omega$. From the example above we know that F need not be S-integrable, so we must modify F to achieve S-integrability.

If is a trivial consequence of the definition of $\Phi(f)$ that it is the limit of the nondecreasing sequence $(\Phi(\min(f, n)) \mid n \in \mathbb{N})$. Moreover, for each $n \in \mathbb{N}$ we know that $\min(F, n)$ is a lifting of $\min(f, n)$. It follows that $\min(F, N)$ is a lifting of f for all infinite $N \in {}^*\mathbb{N}$. We show that for small enough infinite N the function $\min(F, N)$ is S-integrable.

Let $r = \Phi(f) = \lim_{n\to\infty} \Phi(\min(f, n))$. For each $k \geq 1$ choose $n_k \in \mathbb{N}$ so that

$$\big|r - \Phi(\min(f, n_k))\big| < \frac{1}{2k}$$

and choose the sequence so that $n_1 < n_2 < \ldots$. Using the fact that $\min(F, n_k)$ is a lifting of $\min(f, n_k)$, the fact that standardly bounded $\mathcal{A}$-simple functions like $\min(F, n_k)$ are S-integrable, and the previous corollary we have

$$\big|r - \int \min(F, n_k)\, d\mu\big| < \frac{1}{k}$$

for all $k \geq 1$. By overspill there exists an infinite $K \in {}^*\mathbb{N}$ such that

$$\big|r - \int \min(F, {}^*n_K)\, d\mu\big| < \frac{1}{K} \approx 0.$$

Letting $G = \min(F, {}^*n_K)$ we see that G is a lifting of f and $\min(G, n) = \min(F, n)$ is a lifting of $\min(f, n)$ for all standard $n \in \mathbb{N}$. Therefore

$$\mathrm{st}\,\Big(\int G\, d\mu\Big) = r = \lim_{n\to\infty} \Phi(\min(f, n)) = \lim_{n\to\infty} \mathrm{st}\,\Big(\int \min(G, n)\, d\mu\Big).$$

Using condition (1) in Theorem 9.4 this shows that G is S-integrable, which completes the proof. $\square$