

# NONSTANDARD ANALYSIS

MWF 3:00  
243 Altgeld

Math 595 NA

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Nonstandard Analysis (NSA) is a framework for systematically applying some of the basic ideas of model theory to all areas of mathematics. It is especially effective in analysis, geometry, topology, and related areas of mathematics where the concept of *limit* is central.

Nearly fifty years ago, the logician Abraham Robinson observed that the construction of *nonstandard extensions* could provide a rigorous foundation for the use of infinitesimals in basic analysis.<sup>1</sup> Since then, applications of this set of ideas have spread through all of mathematics, greatly extending Robinson's original use of infinitely small and infinitely large numbers, and NSA has become an active branch of research in its own right.

In order to reach advanced applications of NSA in this course, we will assume a knowledge of first-order logic extending at least through the compactness theorem. Students should be able to formulate mathematical statements within first-order logic and should have some experience with nonstandard models. We will also use some tools (such as the construction of *saturated models*) from the beginning parts of model theory.

After developing the basic framework of NSA we will give a substantial indication of how NSA is developed within two areas of advanced mathematics:

- probability and stochastic analysis (based on the Loeb measure construction);
- geometry and functional analysis (based on the nonstandard hull construction).

*Prerequisites:* A knowledge of first-order logic through the compactness theorem; what is covered in the first half of Math 570 at UIUC or in a good undergraduate course in logic will be sufficient.

*References:* There will be no text for the course, and a set of class notes will be distributed during the semester.

The following textbooks give an introduction to NSA as well as some indication of the range of its applications:

- *Lectures on the Hyperreals; an Introduction to Nonstandard Analysis*, by Robert Goldblatt, Springer-Verlag, Graduate Texts in Mathematics 188, 1998.
- *An Introduction to Nonstandard Real Analysis*, by Albert Hurd and Peter Loeb, Academic Press, 1985; reprinted 1998.

The following collections of articles give an introduction to NSA as well as a good indication of the range of applications:

- *Nonstandard Analysis: Theory and Applications*, ed. by Leif Arkeryd, Nigel Cutland, and C. Ward Henson, NATO ASI Series C Vol. 493, Kluwer, 1997.

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<sup>1</sup>A. Robinson, Non-standard analysis, *Proc. Royal Acad. Amsterdam Ser. A*, **64** (1961), 432–440.

- *Nonstandard Analysis and Its Applications*, ed. by Nigel Cutland, London Math. Soc. Student Texts, 10, Cambridge University Press, 1988.
- *Nonstandard Analysis for the Working Mathematician*, ed. by Peter Loeb and Manfred Wolff, Kluwer, 2000.

An older book that covers the basic foundations of NSA (superstructures, etc) in a way closest to what will be done in this course is:

- *Applied Nonstandard Analysis*, by Martin Davis, John Wiley & Sons, 1976. (Out of print and very out of date in respect to the applications that are presented; however, its treatment of basic material, including some things from operator theory, is nicely done.)

Two other books of general interest as background to NSA and its development by Abraham Robinson are:

- *Non-standard Analysis*, by Abraham Robinson; originally published by North-Holland in 1966; reprinted in an inexpensive paperback by Princeton University Press in their Landmarks in Mathematics series, 1996.
- *Abraham Robinson; the Creation of Nonstandard Analysis, a Personal and Mathematical Odyssey*, by Joseph Dauben, Princeton Univ. Press, 1995. Available as a paperback. (A comprehensive biography of Robinson.)

All of these books will be on reserve in the Math Library.

*Grading:* This is primarily a lecture course. There will be regular assignments of homework problems, on the understanding that such work is essential to learning any area of mathematics. Students will be encouraged to give a lecture on a topic or project of their choice.

*Professor Henson's contact information:* 310 Altgeld Hall; 333-2768 (ofc); 365-0024 (home); henson@math.uiuc.edu (email); www.math.uiuc.edu/~henson/ (webpage).

Office Hours: by appointment. Email is recommended for making appointments, asking questions, etc.