Continuous model theory and Gurarij's universal homogeneous separable Banach space

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(includes joint work with Itaï Ben Yaacov)

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We take the signature for unit balls of Banach spaces to be $\mathcal{L} = \{0, c_{r,s}, \| \ \|\}$ where r, s range over rational scalars such that $|r| + |s| \leq 1$. This is a countable signature.

- 0 is a constant symbol.
- $c_{r,s}$ are binary function symbols.
  - Interpret $c_{r,s}(x, y)$ as $rx + sy$.
- $\| \ \|$ is a unary predicate symbol.
- the functions and predicate are 1-Lipschitz in each variable.

It is not difficult to axiomatize the class of (unit balls) of Banach spaces. The least trivial axioms are the ones expressing properties such as

$$\forall x(\|x\| \leq \tfrac{1}{2} \Rightarrow \exists y(x = \tfrac{1}{2}y))$$

In continuous model theory, the best one can do when trying to express this is

$$\sup_x \min(\tfrac{1}{2} \dot{-} \|x\|, \inf_y \|x - \tfrac{1}{2}y\|) = 0$$

and it takes a small analysis argument, using the other axioms as well as the requirement that structures be metrically complete, to show that this condition does what is desired.

Notation

We let T_b denote a theory axiomatizing the class of all (unit balls of) Banach spaces.

- T admits quantifier elimination if for every formula $\varphi(\vec{x})$ and every $\epsilon > 0$ there is a quantifier-free formula $\psi(\vec{x})$ such that the following condition holds in all models of T:
$$\sup_{\vec{x}} \left|\varphi(\vec{x}) - \psi(\vec{x})\right| \leq \epsilon$$
- T is model complete if every embedding between models of T is an elementary embedding.
- T^* is a model companion of T if they have the same signature, T^* is model complete, and T, T^* have the same substructures of models.

- $\mathcal{M} \models T$ is an existentially closed (e.c.) model of T if for every $\mathcal{M} \subseteq \mathcal{N} \models T$, every quantifier-free formula $\varphi(\vec{x}, \vec{y})$, and every $\vec{a} \in M^m$, $\quad \inf_{\vec{y}} \varphi^{\mathcal{M}}(\vec{a}, \vec{y}) = \inf_{\vec{y}} \varphi^{\mathcal{N}}(\vec{a}, \vec{y})$.
- T is inductive if its class of models is closed under unions of chains.

Proposition

Let T be an inductive theory.
(a) T^* is a model companion of T if and only if the models of T^* are exactly the e.c. models of T.
(b) In particular, T has a model companion if and only if the class of e.c. models of T is axiomatizable.

types

Types and type spaces are defined as follows:

- $\mathrm{tp}_{\mathcal{M}}(\vec{a}) = \{\varphi(\vec{x}) \mid \varphi^{\mathcal{M}}(\vec{a}) = 0\}$
- $\mathrm{S}_n(T) = \{\mathrm{tp}_{\mathcal{M}}(\vec{a}) \mid \mathcal{M} \models T \text{ and } \vec{a} \in M^n\}$
- $\widehat{\varphi}(\mathrm{tp}_{\mathcal{M}}(\vec{a})) =$ the unique $r \in \mathbb{R}$ such that $|\varphi - r|^{\mathcal{M}}(\vec{a}) = 0$
- (T complete) The logic topology on $\mathrm{S}_n(T)$ is the coarsest topology making every $\widehat{\varphi}$ continuous.
- (T complete) The induced metric on $\mathrm{S}_n(T)$ is defined by $d(p, q) = \inf\{d(\vec{a}, \vec{b}) \mid \mathcal{M} \models T, \vec{a}, \vec{b} \in M^n, \vec{a} \models p, \text{ and } \vec{b} \models q\}$

isolated types, atomic models

Definition

- A type $p \in S_n(T)$ is isolated (i.e., principal) if it has the same filter of neighborhoods in the logic topology as in the induced metric topology.
- A structure $\mathcal{M}$ is atomic if every type realized in $\mathcal{M}$ is isolated (w.r.t. $T = \mathrm{Th}(\mathcal{M})$).

Theorem (part of the Omitting Types Theorem)

Let T be complete with countable signature, and $p \in S_n(T)$. TFAE:
(1) p is isolated.
(2) p is realized in every model of T.

separable models

Let T be complete with countable signature.

Theorem (Atomic models)

(a) T has an atomic model iff for each $n \in \mathbb{N}$, the isolated types in $S_n(T)$ are dense for the logic topology.
(b) T has at most one separable atomic model.
(c) If $S_n(T)$ is separable under the induced metric, for all $n \in \mathbb{N}$, then T has a separable atomic model.

Theorem (Separable categoricity)

TFAE:
(1) T has exactly one separable model.
(2) Every type in $S_n(T)$ is isolated, for all $n \in \mathbb{N}$.
(3) The induced metric topology is compact on $S_n(T)$, for all $n \in \mathbb{N}$.

Definition (introduced by V I Gurarij in the mid 1960s)

A Banach space X will be said to have the Gurarij property if for every $S\colon E \to X$, every $F \supseteq E$, and every $\lambda > 1$, there is $T\colon F \to X$ such that T extends S, $\|T\| \leq \lambda\|S\|$ and $\|T^{-1}\| \leq \lambda\|S^{-1}\|$.
(Here E, F are finite dimensional and S, T are injective linear.)

Definition

A Gurarij space is a separable Banach space with the Gurarij property.

A few facts are reasonably obvious:

- A Banach space X has the Gurarij property if and only if the unit ball of X is e.c. as a model of T_b.
- separable Banach spaces with the Gurarij property are λ-isomorphic for all $\lambda > 1$.
- every Banach space embeds in one with the Gurarij property.

Tools of Banach space geometry in the late 1960s were adequate to prove:

- If X has the Gurarij property, then the dual space X^* is linearly isometric to $L_1(\mu)$ for some measure μ and X contains subspaces linearly isometric to $\ell_\infty(n)$ for every $n \in \mathbb{N}$.
- No Gurarij space can satisfy the condition in the definition with $\lambda = 1$.

Notation

Let $\mathcal{EC}(T_b)$ be the class of e.c. models of T_b. As noted above, $\mathcal{EC}(T_b)$ is also the class of (unit balls of) Banach spaces with the Gurarij property.

Proposition

The class $\mathcal{EC}(T_b)$ is axiomatizable. Therefore, T_b has a model companion, which we will denote T_b^*; the models of T_b^* are exactly the members of $\mathcal{EC}(T_b)$.

Corollary (easy consequences of the remarks above)

The model companion T_b^* is complete and has quantifier elimination.

QE for T_b^* implies that its types have a natural description.

Consider $\mathcal{M} \models T_b^*$ and $\vec{a} \in M^n$. We may identify $p = \mathrm{tp}_{\mathcal{M}}(\vec{a})$ with the function $t_p \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$t_p(\vec{r}) = \| \sum r_i a_i \|.$$

Indeed, t_p is determined by its restriction to the set

$$\{\vec{r} \in \mathbb{R}^n \mid \sum |r_i| = 1\}.$$

The key to writing axioms for the class $\mathcal{EC}(T_b)$ as well as to verifying more subtle properties of T_b^* is the following "optimal" amalgamation property of Banach spaces:

Lemma

Let X, Y be Banach spaces and I an index set; consider families $\vec{a} = (a_i)_{i\in I} \in X^I, \vec{b} = (b_i)_{i\in I} \in Y^I,$ and $\vec{\varepsilon} = (\varepsilon_i)_{i\in I} \in (\mathbb{R}^{\geq 0})^I$. Also let $\mathbb{R}^{(I)}$ denote the set of all families in $\mathbb{R}^I$ in which only finitely many coordinates are nonzero.
The following conditions are equivalent.

(1) There is a seminorm $\|\cdot\|$ on $X \oplus Y$ that agrees with the given norms on X and Y and satisfies $\|a_i - b_i\| \leq \varepsilon_i$ for all $i \in I$.

(2) For all $\vec{r} \in \mathbb{R}^{(I)}$ one has $\left|\|\sum r_i a_i\| - \|\sum r_i b_i\|\right| \leq \sum |r_i|\varepsilon_i$.

The preceding Lemma gives a precise formula for the induced metric on $S_n(T_b^*)$:

$$d(\mathrm{tp}(\vec{a}), \mathrm{tp}(\vec{b})) = \sup\{\left|\|\sum_{i=1}^n r_i a_i\| - \|\sum_{i=1}^n r_i b_i\|\right| \mid \sum_{i=1}^n |r_i| = 1\}$$

Corollary

For each $n \in \mathbb{N}$, the space $S_n(T_b^*)$ is compact under the induced metric. Therefore T_b^* is separably categorical. Hence there is exactly one separable Banach space with the Gurarij property, up to isometry. In other words, the Gurarij space is unique.

Uniqueness of the Gurarij space was first proved in the mid 1970s by W Lusky. A very elementary proof was given recently by Kubiś and Solecki. One can also see this uniqueness result as a consequence of the Fraïssé construction for metric structures.

Notation

Let $\mathbb{G}$ denote the Gurarij space and $\mathbb{B}$ its unit ball. Consider a tuple $\vec{a} \in \mathbb{B}^m$ and let

- $T_b^*(\vec{a}) = \mathrm{Th}(\mathbb{B}, a_1, \ldots, a_m)$, and
- $\mathrm{S}_n(\vec{a}) = \mathrm{S}_n(T_b^*(\vec{a}))$

Note that $T_b^*(\vec{a})$ and $\mathrm{S}_n(\vec{a})$ only depend on the quantifier-free type of $\vec{a}$, because T_b^* admits quantifier elimination.

Suppose $p \in \mathrm{S}_n(\vec{a})$ is realized by $\vec{b}$; then p can be identified with the function

$$t_p(\vec{r}, \vec{s}) = \| \sum r_i a_i + \sum s_j b_j \|$$

where $\vec{r}, \vec{s}$ come from $\mathbb{R}$ and $\sum |s_j| = 1$.

Moreover, if $p, q \in \mathrm{S}_n(\vec{a})$ are realized by $\vec{b}, \vec{c}$, then

$$d(p, q) = \sup\{|t_p(\vec{r}, \vec{s}) - t_q(\vec{r}, \vec{s})| \mid \vec{r}, \vec{s} \text{ come from } \mathbb{R} \text{ and } \sum |s_j| = 1\}$$

Theorem

Suppose $\vec{a} \in \mathbb{B}^m$ and the linear span of $\vec{a}$ in $\mathbb{G}$ is polyhedral.

- *For each $n \in \mathbb{N}$, the space $\mathrm{S}_n(\vec{a})$ is separable in the induced metric.*
- *Therefore $T_b^*(\vec{a})$ has a separable atomic model, which can be taken to be of the form $(\mathbb{B}, \vec{b})$.*
- *The set of all $\vec{b} \in \mathbb{B}^m$ for which $(\mathbb{B}, \vec{b})$ is an atomic model of $T_b^*(\vec{a})$ is a full orbit under the action of the automorphism group of $\mathbb{G}$.*

The next slides describe joint work with Itaï Ben Yaacov

We obtained an intrinsic geometric characterization of the isolated types in $S_n(\vec{a})$, when the linear span of $\vec{a} \in \mathbb{B}^m$ in $\mathbb{G}$ is polyhedral, and gave easier proofs that atomic models exist in this situation:

Theorem

Let $\vec{a} \in \mathbb{B}^m$ have polyhedral linear span E in $\mathbb{G}$. Consider $\vec{b} \in \mathbb{B}^n$ and let $p = \mathrm{tp}(\vec{b}/\vec{a}) \in S_n(\vec{a})$. Let F be the linear span of $\vec{a}\vec{b}$ in $\mathbb{G}$. The following are equivalent:

(1) p is an isolated type in $S_n(\vec{a})$.

(2) Every linear functional on E has a unique extension to F of the same norm.

Corollary

Let $\vec{a} \in \mathbb{B}^m$ have polyhedral linear span E in $\mathbb{G}$. The following are equivalent:
(1) $(\mathbb{B}, \vec{a})$ is an atomic model of $T_b^*(\vec{a})$.
(2) Every linear functional on E has a unique extension to $\mathbb{G}$ of the same norm.
Moreover, if E is any finite dimensional, polyhedral Banach space, then there exists an isometric linear embedding S of E into $\mathbb{G}$ such that every linear functional on $S(E)$ has a unique extension to $\mathbb{G}$ of the same norm.

Let E be a finite dimensional subspace of $\mathbb{G}$. Let $\mathcal{S}$ be the set of all isometric linear embeddings of E into $\mathbb{G}$; it is a Polish space on which the automorphism group of $\mathbb{G}$ acts continuously.

Theorem

Suppose E is a polyhedral finite dimensional Banach space. The set of $S \in \mathcal{S}$ such that every linear functional on $S(E)$ has a unique extension to $\mathbb{G}$ of the same norm is a dense G_δ set in $\mathcal{S}$ and a full orbit under the action of the automorphism group of $\mathbb{G}$ on $\mathcal{S}$.

In particular, when $m = 1$ and $a \in \mathbb{G}$ satisfies $\|a\| = 1$, we have the result (indicated by Lusky at the end of his uniqueness paper) that the set of smooth points of norm 1 in $\mathbb{G}$ is a full orbit under the action of the automorphism group of $\mathbb{G}$. This is a dense G_δ subset of the unit sphere by a result of Mazur.

What if E is not polyhedral?

If $\vec{a} \in \mathbb{B}^m$ and the linear span of $\vec{a}$ in $\mathbb{G}$ is not polyhedral, then $\mathrm{S}_1(\vec{a})$ has density 2^ω (the maximum possible) in the induced metric. We don't know if $T_b^*(\vec{a})$ has an atomic model in this situation, but we are inclined to conjecture that it does not.

If $\vec{a} \in \mathbb{B}^m$ is nonzero, with $m \geq 1$, then the theory $T_b^*(\vec{a})$ is never separably categorical; indeed, it always has infinitely many different separable models. (When the linear span of $\vec{a}$ is not polyhedral, it follows from the preceding paragraph that $T_b^*(\vec{a})$ has 2^ω many different models. When the linear span of $\vec{a}$ is polyhedral, we conjecture that the number of models is also 2^ω, but we can only prove that there are infinitely many.)

Itaï Ben Yaacov and Alexander Usvyatsov, *On d-finiteness in continuous structures*, Fundamenta Mathematicae 194 (2007), 67–88.
[Gives an exposition of theorems about separable models, in section 1.]

V I Gurarij, *Space of universal disposition, isotopic spaces and the Mazur problem on rotations of Banach spaces*, Siberian Math. Journal 7 (1966), 799–806 (translation from Russian).

W Lusky, *The Gurarij spaces are unique*, Archive of Math. 27 (1976), 627–635.