

# Continuous logic and model theory of metric structures

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# Outline

- 1 Structures and syntax
- 2 Semantics
- 3 Ultraproducts
- 4 Theories

Many classes of **metric structures** arising in analysis and geometry are well-behaved model theoretically, although they are not axiomatizable in the classical sense. Examples:

- (unit balls of) Hilbert spaces and other Banach spaces
- probability measure algebras
- (projective spaces of) nonarchimedean valued fields, with valuations in  $\mathbb{R}$

**Continuous logic** is an attempt to apply model theoretic tools to such classes.

## (bounded) metric structures

- A bounded **metric structure**  $\mathcal{M}$  is based on a bounded metric space  $(M, d)$ .

$\mathcal{M}$  is also equipped with distinguished elements, functions (mapping  $M^n$  to  $M$ ), and predicates (mapping  $M^n$  to a bounded interval in  $\mathbb{R}$ , such as  $[0, 1]$ ).

- The functions and predicates must be uniformly continuous.

## Replace $\{T, F\}$ with $[0, 1]$

- The basic idea of continuous logic is: replace the space of truth values  $\{T, F\}$  by a compact interval in  $\mathbb{R}$ , such as  $[0, 1]$ .
- Quantifiers  $\forall x$  and  $\exists x$  are replaced by  $\sup_x$  and  $\inf_x$ .
- Connectives are continuous functions.

# Symbols and signatures

- A **signature**  $\mathcal{L}$  for continuous logic consists of symbols for constants, functions, and predicates, as usual.
  - constant symbols: interpreted as distinguished elements of  $M$ .
  - $n$ -ary function symbols: interpreted as functions  $M^n \rightarrow M$ .
  - $n$ -ary predicate symbols: interpreted as functions  $M^n \rightarrow [0, 1]$ .
- $\mathcal{L}$  specifies a **modulus of uniform continuity** for each function symbol and predicate symbol. (e.g.: 1-Lipshitz.)
- The **metric** is considered as a (logical) binary predicate (exactly as equality is used in classical logic).

## Terms and atomic formulas

- **Terms** of  $\mathcal{L}$  are defined inductively using variables, constant symbols, and function symbols, as usual.
- **Atomic formulas** of  $\mathcal{L}$  are expressions of the form  $P(t_1, \dots, t_n)$ , where  $P$  is an  $n$ -ary predicate symbol of  $\mathcal{L}$  and  $t_1, \dots, t_n$  are terms of  $\mathcal{L}$ .

## Example: probability algebras

The signature of probability algebras is  $\mathcal{L}_{Pr} = \{0, 1, \cdot^c, \cup, \cap, \mu\}$ .

- $0, 1$  are constant symbols.
- $\cdot^c$  (complement) is a unary function symbol.
- $\cup, \cap$  (union, intersection) are binary function symbols.
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Thus:

- $z, x \cap y^c, x \Delta y$  are **terms** (values in the algebra).
- $\mu(x), \mu(x \cap y^c)$  are **atomic formulas** (values in  $[0, 1]$ ).

# Formulas

The **formulas** of a continuous signature  $\mathcal{L}$  are built inductively starting from the atomic formulas of  $\mathcal{L}$ , as follows:

- If  $\varphi_1, \dots, \varphi_m$  are formulas and  $u: [0, 1]^m \rightarrow [0, 1]$  is continuous, then  $u(\varphi_1, \dots, \varphi_m)$  is a formula.
- If  $\varphi$  is a formula and  $x$  is a variable, then  $\sup_x \varphi$  and  $\inf_x \varphi$  are formulas.

# Probability algebras, continued.

- Some useful connectives:

$$\begin{array}{cc} \min(a, b) & \max(a, b) \\ |a - b| & a \dot{-} b \end{array}$$

(Here we define  $a \dot{-} b := \max(a - b, 0)$ .)

- Thus (in  $\mathcal{L}_{Pr}$ )  $|\mu(x) - \mu(y)|$  is a **quantifier-free** formula, and
- $\sup_x \inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)|$  is a formula (it is in fact a **sentence**, since it has no **free variables**).

## Equality...?

In classical logic the symbol  $=$  always satisfies:

$$x = x \qquad (x = y) \rightarrow ((x = z) \rightarrow (y = z)) \qquad (\text{ER})$$

Replacing " $x = y$ " with " $d(x, y)$ " and " $\varphi \rightarrow \psi$ " with " $(\psi \div \varphi)$ ":

$$d(x, x) = 0 \qquad ((d(y, z) \div d(x, z)) \div d(x, y)) = 0$$

That is,  $d$  is a pseudo-metric:

$$d(x, x) = 0 \qquad d(y, z) \leq d(x, z) + d(x, y) \qquad (\text{PM})$$

Similarly, in classical logic  $=$  is a congruence relation:

$$(x = y) \rightarrow (P(x, \bar{z}) \rightarrow P(y, \bar{z})) \quad (\text{CR})$$

This translates to:

$$\left( (P(y, \bar{z}) \dot{-} P(x, \bar{z})) \dot{-} d(x, y) \right) = 0$$

which says that  $P$  is 1-Lipschitz:

$$P(y, \bar{z}) - P(x, \bar{z}) \leq d(x, y). \quad (1L)$$

So, a literal translation of these statements from classical logic requires all predicate and function symbols to be 1-Lipschitz in  $d$ . (We can relax this to “uniformly continuous”.)

# $\mathcal{L}$ -Structures

## Definition

An  $\mathcal{L}$ -pre-structure  $\mathcal{M}$  is a set  $M$ , equipped with a pseudo-metric  $d^{\mathcal{M}}$  and interpretations  $c^{\mathcal{M}}, f^{\mathcal{M}}, P^{\mathcal{M}}$  of all symbols  $c, f, P \in \mathcal{L}$  such that every  $f^{\mathcal{M}}$  and  $P^{\mathcal{M}}$  satisfies the modulus specified by  $\mathcal{L}$ .  $\mathcal{M}$  is an  $\mathcal{L}$ -structure if, in addition,  $d^{\mathcal{M}}$  is a complete metric.

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- In classical logic  $=^{\mathcal{M}}$  is a congruence relation; thus the quotient of  $M$  by  $=^{\mathcal{M}}$  is well-defined and it cannot be distinguished from  $\mathcal{M}$  by the logic.
- Similarly, in continuous logic a prestructure  $\mathcal{M}$  is logically indistinguishable from the completion  $\widehat{\mathcal{N}}$  of the quotient  $\mathcal{N} = \mathcal{M}/\sim_d$ .  
 $(a \sim_d b \iff d(a, b) = 0)$
- We call  $\widehat{\mathcal{M}/\sim_d}$  the **completion** of the pre-structure  $\mathcal{M}$ .

# Probability algebras, redux

- Let  $(\Omega, \mathfrak{B}, \mu)$  be a probability space.
- Let  $\mathfrak{I}_0 \leq \mathfrak{B}$  be the ideal of  $\mu$ -null sets, and  $\widehat{\mathfrak{B}} = \mathfrak{B}/\mathfrak{I}_0$ . Then  $\widehat{\mathfrak{B}}$  is a Boolean algebra and  $\mu$  induces  $\widehat{\mu}: \widehat{\mathfrak{B}} \rightarrow [0, 1]$ . The pair  $(\widehat{\mathfrak{B}}, \widehat{\mu})$  is a **probability algebra**.
- $\widehat{\mathfrak{B}}$  admits a complete metric:  $d(a, b) = \widehat{\mu}(a \Delta b)$ .
- $\widehat{\mu}$  and the Boolean operations are 1-Lipschitz.

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- $\widehat{\mathfrak{B}}$  admits a complete metric:  $d(a, b) = \widehat{\mu}(a \Delta b)$ .
- $\widehat{\mu}$  and the Boolean operations are 1-Lipschitz.
- $(\mathfrak{B}, 0, 1, \cap, \cup, \cdot^c, \mu)$  is a pre-structure;  
 $(\widehat{\mathfrak{B}}, 0, 1, \cap, \cup, \cdot^c, \widehat{\mu})$  is its completion (in particular, it is a structure).

# Semantics

As usual, the notation  $\varphi(x_1, \dots, x_n)$  [or simply  $\varphi(\bar{x})$ ] means that the free variables of  $\varphi$  are among  $x_1, \dots, x_n$ , which are distinct.

- If  $\mathcal{M}$  is a structure and  $\bar{a} \in M^n$ , we define the (truth) value  $\varphi^{\mathcal{M}}(\bar{a}) \in [0, 1]$  inductively, in the “obvious way”.

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## Example

Let  $\mathcal{M} = (M, 0, 1, \cdot^c, \cup, \cap, \mu)$  be a probability algebra, and take  $\varphi(x)$  to be the formula  $\inf_y |\mu(x \cap y) - \frac{1}{2}\mu(x)|$ .

- If  $a \in M$  is an atom, then  $\varphi^{\mathcal{M}}(a) = \frac{1}{2}\mu(a)$ .
- If  $a$  has no atoms below it then  $\varphi^{\mathcal{M}}(a) = 0$ .

## Various “elementary” notions

- **Elementary equivalence** (denoted  $\mathcal{M} \equiv \mathcal{N}$ ): If  $\mathcal{M}, \mathcal{N}$  are two structures then  $\mathcal{M} \equiv \mathcal{N}$  if  $\varphi^{\mathcal{M}} = \varphi^{\mathcal{N}} \in [0, 1]$  for every sentence  $\varphi$ .  
Equivalently:  $\varphi^{\mathcal{M}} = 0 \iff \varphi^{\mathcal{N}} = 0$  for all sentences  $\varphi$ .
- **Elementary extension** (denoted  $\mathcal{M} \preceq \mathcal{N}$ ): This holds if  $\mathcal{M} \subseteq \mathcal{N}$  and  $\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(\bar{a})$  for every formula  $\varphi(\bar{x})$  and  $\bar{a} \in M$ . It implies  $\mathcal{M} \equiv \mathcal{N}$ .

### Lemma (Elementary chains)

*The union of an elementary chain  $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \dots$  is an elementary extension of each  $\mathcal{M}_i$ .*

Caution: by the **union** of an increasing chain we mean the completion of its set-theoretic union.

# Ultraproducts

- $(\mathcal{M}_i \mid i \in I)$  are  $\mathcal{L}$ -structures,  $\mathcal{U}$  an ultrafilter on  $I$ .
- We let  $N_0 = \prod_{i \in I} M_i$  as a set.
- We interpret the symbols:

$$\begin{aligned} c^{\mathcal{N}_0} &= (c^{\mathcal{M}_i} \mid i \in I) \\ f^{\mathcal{N}_0}((a_i \mid i \in I), \dots) &= (f^{\mathcal{M}_i}(a_i, \dots) \mid i \in I) \in N_0 \\ P^{\mathcal{N}_0}((a_i \mid i \in I), \dots) &= \lim_{i, \mathcal{U}} P^{\mathcal{M}_i}(a_i, \dots) \in [0, 1] \end{aligned}$$

- This makes  $\mathcal{N}_0$  a pre-structure. We define the **ultraproduct** to be  $\widehat{\mathcal{N}}_0$  (the completion) and denote it by  $\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ .
- The image of  $(\bar{a}) \in N_0$  in  $\widehat{\mathcal{N}}_0$  is denoted  $(\bar{a})_{\mathcal{U}}$ ; note that

$$(\bar{a})_{\mathcal{U}} = (\bar{b})_{\mathcal{U}} \iff 0 = \lim_{i, \mathcal{U}} d(a_i, b_i) \quad \left[ = d^{\mathcal{N}_0}((\bar{a}), (\bar{b})) \right].$$

# Properties of ultraproducts

- **Łoś's Theorem:** for every formula  $\varphi(x, y, \dots)$  and elements  $(\bar{a})_{\mathcal{U}}, (\bar{b})_{\mathcal{U}}, \dots \in \mathcal{N} = \prod \mathcal{M}_i / \mathcal{U}$ :

$$\varphi^{\mathcal{N}}((\bar{a})_{\mathcal{U}}, (\bar{b})_{\mathcal{U}}, \dots) = \lim_{i, \mathcal{U}} \varphi^{\mathcal{M}_i}(a_i, b_i, \dots).$$

- [Easy]  $\mathcal{M} \equiv \mathcal{N}$  ( $\mathcal{M}$  and  $\mathcal{N}$  are elementarily equivalent) if and only if  $\mathcal{M}$  admits an elementary embedding into an ultrapower of  $\mathcal{N}$ .
- [Deeper: generalising Keisler & Shelah]  $\mathcal{M} \equiv \mathcal{N}$  if and only if  $\mathcal{M}$  and  $\mathcal{N}$  have ultrapowers that are isomorphic.

# Theories

- A **theory**  $T$  is a set of sentences (closed formulas).
- $\mathcal{M}$  is a **model** of  $T$  (written  $\mathcal{M} \models T$ ) if
$$\varphi^{\mathcal{M}} = 0 \text{ for all } \varphi \in T.$$
- We sometimes write  $T$  as a set of **conditions** “ $\varphi = 0$ ”.  
We may also allow as conditions things of the form “ $\varphi \leq r$ ”, “ $\varphi \geq r$ ”, “ $\varphi = r$ ”, etc.
- If  $\mathcal{M}$  is any structure then its **theory** is
$$\text{Th}(\mathcal{M}) = \{\varphi \mid \varphi^{\mathcal{M}} = 0\}.$$
Theories of this form are called **complete**.

# Compactness

## Theorem (Compactness)

*A theory is satisfiable if and only if it is finitely satisfiable.*

Note that:

$$T \equiv \{ \text{“}\varphi \leq 2^{-n}\text{”} \mid n < \omega \text{ and “}\varphi = 0\text{”} \in T \}.$$

## Corollary

*Assume that  $T$  is **approximately** finitely satisfiable. Then  $T$  is satisfiable.*

# Löwenheim-Skolem Theorem

Notation:  $\|\cdot\|$  denotes the **metric** density character.

$\mathcal{L}$  is a signature with  $\leq \kappa$  symbols.

$\mathcal{M}$  is an  $\mathcal{L}$ -structure and  $A \subseteq M$  has  $\|A\| \leq \kappa$ .

## Theorem

*There exists an elementary substructure  $\mathcal{N}$  of  $\mathcal{M}$  such that  $A \subseteq N$  and  $\|N\| \leq \kappa$ .*

## Elementary (axiomatizable) classes

A class  $\mathcal{C}$  of  $\mathcal{L}$ -structures is **elementary** or **axiomatizable** if there is an  $\mathcal{L}$ -theory  $T$  such that  $\mathcal{C}$  is the class of all models of  $T$ . When this holds we call  $T$  a set of **axioms** for  $\mathcal{C}$ .

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### Theorem

*Let  $\mathcal{C}$  be a class of  $\mathcal{L}$ -structures. Then  $\mathcal{C}$  is axiomatizable iff  $\mathcal{C}$  is closed under isomorphisms, ultraproducts, and ultraroots.*

Here:  $\mathcal{M}$  is an **ultraroot** of  $\mathcal{N}$  if  $\mathcal{N}$  is isomorphic to some ultrapower of  $\mathcal{M}$ .

# Universal theories

- A theory consisting solely of sentences of the form  $\sup_{\bar{x}} \varphi(\bar{x})$ , where  $\varphi$  is quantifier-free, is called **universal**. Universal theories are those stable under substructures.
- We may write  $(\sup_{\bar{x}} \varphi) = 0$  as  $\forall \bar{x} (\varphi = 0)$ .
- Similarly, we may write  $(\sup_{\bar{x}} |\varphi - \psi|) = 0$  as  $\forall \bar{x} (\varphi = \psi)$ .
- And if  $\sigma, \tau$  are terms: we may write  $(\sup_{\bar{x}} d(\sigma, \tau)) = 0$  as  $\forall \bar{x} (\sigma = \tau)$ .

# The (universal) theory of probability algebras

The class of probability algebras is axiomatized by the following set  $Pr$  of conditions:

*The equational axioms for Boolean algebras*

$$\forall xy \left( d(x, y) = \mu(x \Delta y) \right)$$

$$\forall xy \left( \mu(x) + \mu(y) = \mu(x \cap y) + \mu(x \cup y) \right)$$

$$\mu(1) = 1$$

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The **model companion** of  $Pr$  is the  $\forall\exists$ -theory  $APr$  of **atomless** probability algebras, consisting of  $Pr$  together with:

$$\sup_x \inf_y \left| \mu(x \cap y) - \frac{1}{2} \mu(x) \right| = 0.$$

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The theory  $APr$  is  $\omega$ -categorical, and it is therefore complete; it also satisfies quantifier elimination.

## Probability algebras with an automorphism

An interesting research topic concerns the metric structures  $(\mathcal{M}, \tau)$  where  $\mathcal{M}$  is a probability algebra and  $\tau$  is an automorphism of  $\mathcal{M}$ . These all arise from measure preserving automorphisms of a probability space.

The fact that  $\tau$  is an automorphism (of a model of  $Pr$ ) can be axiomatized in a suitable signature  $\mathcal{L}_{Pr, \tau}$  extending  $\mathcal{L}_{Pr}$ , by a theory  $Pr_{\tau}$ . For example

$$\sup_x |\mu(\tau(x)) - \mu(x)| = 0$$

expresses the fact that  $\tau$  is measure preserving and

$$\sup_y \inf_x d(y, \tau(x)) = 0$$

expresses the fact that  $\tau$  is surjective (given that  $\tau$  is isometric).


Of special interest are the  $(\mathcal{M}, \tau) \models Pr_\tau$  that arise from an **aperiodic** automorphism  $S$  of an atomless probability space  $(\Omega, \mathfrak{B}, \mu)$ ; this means that for each  $n \in \mathbb{N}$  the set  $\{\omega \in \Omega \mid S^n(\omega) = \omega\}$  has measure 0. This property can be axiomatized (over  $Pr_\tau$ ) by the conditions (for  $n \geq 1$ )


$$\inf_x \max(1/n \div \mu(x), \mu(x \cap \tau(x)), \dots, \mu(x \cap \tau^{n-1}(x))) = 0.$$

The theory consisting of  $Pr_\tau$  and these conditions is:  $APr_A$ .




**Theorem.** (Berenstein and WH) The theory  $APr_A$  is the model companion of  $Pr_\tau$ ; it is complete, has quantifier elimination, and is stable. Further (Ben Yaacov),  $APr_A$  is not superstable; thus it is  $\kappa$ -stable iff  $\kappa^\omega = \kappa$ .

## Model theory for metric structures, based on continuous logic

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## Probability algebras

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