Continuous first order logic for metric structures and the nonstandard hull construction

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- The *nonstandard hull* construction is one of the fundamental tools of nonstandard analysis. Essentially every way of constructing standard objects from nonstandard ones has a nonstandard hull construction in the background.

- The nonstandard hull construction is (usually) applied to an *internal metric structure* $\mathcal{M}$.

- It produces an *"ordinary" metric structure* $\widehat{\mathcal{M}}$.

- Continuous logic explains the *relationship* between $\mathcal{M}$ and $\widehat{\mathcal{M}}$, and provides *tools* for making use of this relationship.

1 Structures and syntax

(bounded) metric structures

- A bounded *metric structure* $\mathcal{M}$ is based on a bounded complete metric space (M, d).

  $\mathcal{M}$ is also equipped with distinguished elements, functions (mapping M^n to M), and predicates (mapping M^n to a bounded interval in $\mathbb{R}$, such as $[0, 1]$).

- The functions and predicates must be uniformly continuous.

Replace $\{T, F\}$ with $[0, 1]$

- The basic idea of (bounded) *continuous logic* is: replace the space of truth values $\{T, F\}$ by a compact interval in $\mathbb{R}$, such as $[0, 1]$.

- Quantifiers $\forall x$ and $\exists x$ are replaced by $\sup_x$ and $\inf_x$.

- Connectives are continuous functions.

Symbols and signatures

- A *signature* $\mathcal{L}$ for continuous logic consists of symbols for constants, functions, and predicates, as usual.
  - constant symbols: interpreted as distinguished elements of M.
  - n-ary function symbols: interpreted as functions $M^n \to M$.
  - n-ary predicate symbols: interpreted as functions $M^n \to [0,1]$.
- $\mathcal{L}$ specifies a *modulus of uniform continuity* for each function symbol and predicate symbol.
- The *metric* is considered as a binary predicate
  (exactly as equality is used in classical logic).

Terms and atomic formulas

- *Terms* of $\mathcal{L}$: defined inductively, as usual, using variables, constant symbols, and function symbols of $\mathcal{L}$.

- *Atomic formulas* of $\mathcal{L}$: $P(t_1, \dots, t_n)$, where P is an n-ary predicate symbol of $\mathcal{L}$ and $t_1, \dots, t_n$ are terms of $\mathcal{L}$.

Formulas

The *formulas* of a continuous signature $\mathcal{L}$ are built inductively starting from the atomic formulas of $\mathcal{L}$, as follows:

- If $\varphi_1, \dots, \varphi_m$ are formulas and $u\colon [0,1]^m \to [0,1]$ is continuous, then $u(\varphi_1, \dots, \varphi_m)$ is a formula.

- If φ is a formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are formulas.

$\mathcal{L}$-Structures

Definition. An $\mathcal{L}$-*structure* $\mathcal{M}$ is a set M, equipped with a complete metric $d^{\mathcal{M}}$ (bounded by 1) and interpretations $c^{\mathcal{M}}$, $f^{\mathcal{M}}$, $P^{\mathcal{M}}$ of all symbols $c, f, P \in \mathcal{L}$ such that every $f^{\mathcal{M}}$ and $P^{\mathcal{M}}$ satisfies the modulus specified by $\mathcal{L}$.

$\mathcal{M}$ is an $\mathcal{L}$-*pre-structure* if these requirements are all met except that $d^{\mathcal{M}}$ may be a *pseudo*-metric, or $d^{\mathcal{M}}$ may not be *complete*.

- The completion of the quotient $\mathcal{M}/[d(a,b)=0]$ is an $\mathcal{L}$-structure which we denote by $\widehat{\mathcal{M}}$ and call the *hull* of $\mathcal{M}$.

- It turns out that $\mathcal{M}$ and $\widehat{\mathcal{M}}$ cannot be distinguished in continuous logic.

Probability algebras

- Let $(\Omega, \mathfrak{B}, \mu)$ be a probability space.
- $\mathfrak{B}$ admits a pseudometric: $d(A, B) = \mu(A \triangle B)$.
- Let $\mathfrak{I}_0 \leq \mathfrak{B}$ be the ideal of μ-null sets, and $\widehat{\mathfrak{B}} = \mathfrak{B}/\mathfrak{I}_0$. Then $\widehat{\mathfrak{B}}$ is a Boolean algebra and μ induces $\widehat{\mu} \colon \widehat{\mathfrak{B}} \to [0,1]$. The pair $(\widehat{\mathfrak{B}}, \widehat{\mu})$ is a *probability algebra*.
- $\widehat{\mathfrak{B}}$ admits a complete metric: $d(a, b) = \widehat{\mu}(a \triangle b)$.
- $(\mathfrak{B}, 0, 1, \cap, \cup, \cdot^c, \mu)$ is a pre-structure;
  $(\widehat{\mathfrak{B}}, 0, 1, \cap, \cup, \cdot^c, \widehat{\mu})$ is its hull (in particular, it is a structure).

2 Semantics

Values of formulas

Let $\varphi(\bar{x})$ be an $\mathcal{L}$-formula and $\mathcal{M}$ an $\mathcal{L}$-pre-structure.

- If $\mathcal{M}$ is a structure and $\bar{a} \in M^n$, we define the *value* $\varphi^{\mathcal{M}}(\bar{a})$ inductively, in the "obvious way".
- Each function $\varphi^{\mathcal{M}} \colon M^n \to [0,1]$ is uniformly continuous.
  (Indeed, its modulus is independent of $\mathcal{M}$.)

Various "elementary" notions

Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures (or even pre-structures).

- *Elementary equivalence* (denoted $\mathcal{M} \equiv \mathcal{N}$): $\mathcal{M} \equiv \mathcal{N}$ if $\varphi^{\mathcal{M}} = \varphi^{\mathcal{N}}$ for every $\mathcal{L}$-sentence φ.
- *Elementary embedding*: $F \colon M \to N$ is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$ if $\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(F(\bar{a}))$ holds for every $\mathcal{L}$-formula $\varphi(\bar{x})$ and every tuple $\bar{a}$ from M.
- The natural map from a pre-structure $\mathcal{M}$ to its hull $\widehat{\mathcal{M}}$ is an elementary embedding.

Theories

Fix a continuous signature $\mathcal{L}$.

- An $\mathcal{L}$-*theory* T is a set of $\mathcal{L}$-sentences.
- $\mathcal{M}$ is a *model* of T (written $\mathcal{M} \models T$) if $\mathcal{M}$ is an $\mathcal{L}$-structure and
  $\varphi^{\mathcal{M}} = 0$ for all $\varphi \in T$.
- If $\mathcal{M}$ is any $\mathcal{L}$-pre-structure then its *theory* is
  $\mathrm{Th}(\mathcal{M}) = \{\varphi \mid \varphi \text{ is an } \mathcal{L}\text{-sentence and } \varphi^{\mathcal{M}} = 0\}$.
  Theories of this form are called *complete*.
- A class $\mathcal{C}$ of $\mathcal{L}$-structures is *elementary* or *axiomatizable* if there is an $\mathcal{L}$-theory T such that $\mathcal{C}$ is the class of all models of T. When this holds we call T a set of *axioms* for $\mathcal{C}$.

Nonstandard hulls

Let $\mathcal{M}$ be an *internal $\mathcal{L}$-pre-structure*; that is,
M is an internal set, $d^{\mathcal{M}}$ is an internal pseudometric on M with values in ${}^*[0,1]$, and all the functions $f^{\mathcal{M}}\colon M^n \to M$ and $P^{\mathcal{M}}\colon M^n \to {}^*[0,1]$ satisfy the moduli specified by $\mathcal{L}$.

We construct the *nonstandard hull* of $\mathcal{M}$:

- Step 1: replace $d^{\mathcal{M}}$ and each $P^{\mathcal{M}}$ by its (pointwise) standard part; M and each $c^{\mathcal{M}}$ and $f^{\mathcal{M}}$ are unchanged.
- The result is an $\mathcal{L}$-pre-structure in the ordinary sense, denoted $\mathcal{M}_{\mathrm{st}}$.
- Step 2: form the hull of $\mathcal{M}_{\mathrm{st}}$, denoted $\widehat{\mathcal{M}}$. This is the nonstandard hull of $\mathcal{M}$.

Fact. *For every $\mathcal{L}$-formula $\varphi(\bar{x})$ and every tuple $\bar{a}$ from M,*

$$\varphi^{\widehat{\mathcal{M}}}(\bar{a}) = \varphi^{\mathcal{M}_{\mathrm{st}}}(\bar{a}) = \mathrm{st}\left(\varphi^{\mathcal{M}}(\bar{a})\right)$$

Fact. *Assume the nonstandard extension satisfies the κ-isomorphism property and that the number of nonlogical symbols of $\mathcal{L}$ is $< \kappa$.*
Let $\mathcal{M}, \mathcal{N}$ be internal $\mathcal{L}$-pre-structures. Then

$$\widehat{\mathcal{M}} \equiv \widehat{\mathcal{N}} \quad \Longrightarrow \quad \widehat{\mathcal{M}} \cong \widehat{\mathcal{N}}$$

That is, Th($\widehat{\mathcal{M}}$) determines $\widehat{\mathcal{M}}$ up to isomorphism.

3 Probability algebras

Theories of probability algebras

The class of probability algebras is axiomatized by the following set Pr of conditions:

The equational axioms for Boolean algebras

$$\sup_x \sup_y \left| d(x, y) - \mu(x \triangle y) \right| = 0$$

$$\sup_x \sup_y \left| (\mu(x) + \mu(y)) - (\mu(x \cap y) + \mu(x \cup y)) \right| = 0$$

$$\mu(1) = 1$$

The class of *atomless* probability algebras is axiomatized by the set APr of conditions, which consists of Pr together with:

$$\sup_x \inf_y \left| \mu(x \cap y) - \tfrac{1}{2}\mu(x) \right| = 0.$$

Properties of APr

- APr is complete and has quantifier-elimination;
- it is the model companion of Pr; thus atomless probability algebras are the existentially closed probability algebras;
- it is ω-stable, and its model-theoretic independence relation is (conditional) independence in the sense of probability:

  $$A \mathop{\smile\!\!\!\!\!\!\,|}_C B \iff \mathbb{E}(a \mid BC) = \mathbb{E}(a \mid C) \text{ for all } a \in A.$$

  Here A, B, C are subsets of a model of APr.

Probability algebras with an automorphism

We consider the metric structures $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a probability algebra and τ is an automorphism of $\mathcal{M}$. These all arise from measure preserving automorphisms of a probability space. The class of such structures is axiomatizable.

For example

$$\sup_x \left| \mu(\tau(x)) - \mu(x) \right| = 0$$

expresses the fact that τ is measure preserving and

$$\sup_y \inf_x d(y, \tau(x)) = 0$$

expresses the fact that τ is surjective (given that τ is isometric).

Let Pr_τ denote a theory axiomatizing these structures.

Of special interest are the $(\mathcal{M}, \tau) \models Pr_\tau$ that arise from an *aperiodic* automorphism S of an *atomless* probability space $(\Omega, \mathfrak{B}, \mu)$; "aperiodic" means that for each $n \in \mathbb{N}$ the set $\{\omega \in \Omega \mid S^n(\omega) = \omega\}$ has measure 0. Using Rokhlin's Lemma, this property can be axiomatized (over $APr \cup Pr_\tau$) by the conditions (for $n \geq 1$)

$$\inf_x \max\left(1/n \dot{-} \mu(x), \mu(x \cap \tau(x)), \ldots, \mu(x \cap \tau^{n-1}(x))\right) = 0.$$

($u \dot{-} v = \max(u - v, 0)$ so $u \dot{-} v \leq \delta$ iff $u - \delta \leq v$.)

Let APr_A denote the theory obtained by adding these conditions to $APr \cup Pr_\tau$.

Properties of APr_A

- APr_A is complete and has quantifier-elimination;
- it is the model companion of Pr_τ; thus its models are the existentially closed probability algebras with an automorphism;
- it is stable, and its model-theoretic independence relation is conditional independence applied to orbits under the automorphism;
- but it is not superstable, and thus it is κ-stable iff $\kappa^\omega = \kappa$.

This is mostly joint work with Alex Berenstein; the last fact is due to Itaï Ben Yaacov, as are the initial results about APr.

Nonstandard hulls of measure preserving automorphisms of Loeb probability spaces

Now we work in a nonstandard extension that has the ω_1-isomorphism property, and we apply the completeness of APr_A.

First consider $(X, \mathcal{A}, \mu)$ in which $X = \{x \in {}^*\mathbb{N} \mid 1 \leq x \leq H\}$, with H infinite, $\mathcal{A} = {}^*\mathcal{P}(X)$, and $\mu(A) \equiv \frac{1}{H}|A|$. Further, let $S\colon X \to X$ be defined by $S(x) \equiv x + 1 \pmod{H}$. The nonstandard hull $\widehat{\mathcal{M}}_H$ of $\mathcal{M}_H = (\mathcal{A}, \mu, S)$ is obviously a model of APr_A; that is, the probability algebra is atomless and S is aperiodic.

Next consider $(Y, \mathcal{B}, \nu)$ any internal probability space such that the atoms of $\mathcal{B}$ all have infinitesimal measure, and $T\colon Y \to Y$ internal and measure preserving, such that $\{x \in X \mid T^n(x) = x\}$ has infinitesimal measure for each standard n. Then
the nonstandard hull of $(\mathcal{B}, \nu, T)$ is always isomorphic to $\widehat{\mathcal{M}}_H$.

Model theory for metric structures, based on continuous logic

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