OVERVIEW OF THE AIM WORKSHOP "MODEL THEORY OF METRIC STRUCTURES" SEPT. 18–22, 2006

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Background. First-order logic fails to properly describe classes of metric structures. Over the years, several independent approaches sought to overcome this difficulty by developing alternative logics or by using natural analogues of model theoretic methods. Among these we may mention: the ultraproduct construction of Dacunha-Castelle and Krivine [DCK72], which became a widely used tool in the geometry of Banach spaces; Fajardo and Keisler's work on stochastic processes and adapted spaces [FK02]; Henson's logic of positive bounded formulas and approximate satisfaction for Banach space structures [Hen76, HI02] and Iovino's work on stability in this framework [Iov99]; and Ben-Yaacov's framework of compact abstract theories [Ben05a] specialized to the setting of metric structures [Ben05b].

Recently a new formalism emerged for the model theory of metric structures, in the form of *continuous first-order logic*, to which all these paths seem to converge. This is an improved variant of a logic whose model theory was studied by Chang and Keisler [CK66]; it can be briefly described as first-order logic with the following modifications:
– The set of possible truth values for predicates (and therefore for formulas) is the interval $[0, 1]$, rather than $\{T, F\}$. As a consequence, the (boolean) equality relation is replaced with a continuous *distance* relation (i.e., a $[0, 1]$-valued metric).
– Logical connectives are continuous mappings from Cartesian powers of $[0, 1]$ to $[0, 1]$.
– The quantifiers are $\sup_x \varphi$ and $\inf_x \varphi$, with the obvious semantics.
Two extensive treatments of continuous first-order logic have been written in the last year; see [BUa] and [BBHU].

In the brief time since its adoption, continuous first-order logic created a surge of activity in the model theory of metric structures, both theoretical (e.g., local stability, superstability, perturbed categoricity) and application-related (e.g., generic automorphisms of measure algebras, independence in L^p Banach lattices, ample generics in the unitary group).

Continuous first-order logic provides model theorists and analysts with a common language: on the one hand, this is due to its close proximity to first-order logic, and on the other, through its use of familiar analytic constructs (e.g., sup and inf instead of $\forall$ and $\exists$). It is also simpler and more natural than any of the other logics mentioned above.

Themes and goals. This is a timely opportunity to bring together specialists from model theory and the application areas. Combining their expertise, they will be able to explore from new angles certain examples, applications, and theoretical problems that define the frontier of research on continuous first-order logic and the model theory of metric structures. As our goals seem to favor free discussions over a series of lectures, the format of an AIM workshop is ideal for such a meeting.

The main topics on which the workshop would focus are:

Pure model theory. Much of the standard toolbox of model theory and stability theory has been extended to continuous logic. There are still many tools whose generalization is known to be problematic, or for which we are still looking for the right analogue. In addition, continuous logic introduces promising new tools and phenomena, such as the possibility to perturb metric structures, which still require study and exploration. The following two topics are cutting-edge at this time:
– Superstability: It is known that the "obvious" definition of superstability is too strong for continuous logic, and must be weakened in a manner that takes the metric into account (this dates back to [Iov99], revisited in [Ben05b]). This weakening requires us to re-work superstability almost from scratch; while many of the classical proofs fail, examples and results suggest that a (somewhat weaker) general theory can be recovered. One complication of continuous logic is that single elements behave in many respects like infinite tuples of elements in a model of a classical first-order theory. Various definitions of a "weakly finite" element, and conditions of sufficient ubiquity of such elements in the structure, allow us to exclude certain pathologies and further generalize classical results to the metric setting [BUb]. We intend to refine and extend these techniques.
– Perturbations of metric structures: These arise in many situations. For example, the Banach-Mazur distance between Banach spaces is defined through perturbations of the norm; the metric of uniform convergence induces a notion of perturbation of automorphisms of a metric structure. In a recent preprint [Bena], a model-theoretic framework for such notions of perturbation is defined, and theories which have a unique separable model up to arbitrarily small perturbations are characterized.

Applications and examples. Analysis, probability, and metric space geometry represent new application areas for most parts of model theory. Continuous logic seems to be a convenient way to approach such applications and its formalism resonates with how mathematicians from these areas usually express themselves. In our list of tentative participants we include the names of several researchers from these areas whose interest in connections to model theory makes them ready for interdisciplinary work of the kind we have in mind. The following application topics seem to be fruitful for active discussion between these researchers and model theorists.

– A wide-ranging model theoretic study of metric structures coming from all aspects of probability theory [FK02, Benb, BH]. This includes adapted probability spaces as well as actions of groups on probability measure algebras (open problems focus on non-amenable groups such as the nonabelian free groups). The latter topic leads toward dynamics and properties of the automorphism group of the Lebesgue probability algebra. (See also the next topic.)
– Automorphism groups $\mathrm{Aut}(M)$ of highly homogeneous separable metric structures M, such as Urysohn's metric space, Gurarii's Banach space, separable Hilbert spaces and the Lebesgue probability algebra. These structures are analogous to ultrahomogeneous countable structures, such as the random graph. Of interest are topological, dynamic, and descriptive set-theoretic properties of $\mathrm{Aut}(M)$ acting on M. Perturbation of automorphisms (see above) seems to be useful here.

– Other promising (but less well formed) topics include noncommutative probability and unitary representations of C^*-algebras and locally compact groups. By the first topic we mean the model theory of unital von Neumann algebras with a faithful tracial state; suggestively to model theorists, these structures have an ultrapower construction [McD70] and a notion of independence coming from free probability [NSS02].

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