

Problem 1

Let $A = \{0, 1\}$ and $B = \{2, 3\}$. Write down the following sets, using proper mathematical notation (e.g., use parentheses or braces as appropriate).

1. The power set of A : $P(A) =$

Solution: The power set of A is the set of all subsets of A , so $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

Comments. Note that the elements of $P(A)$ are *sets*, so the brace notation is needed. Also, the empty set, \emptyset , is a subset of any set, so it must be included.

2. The cartesian product of A and B : $A \times B =$

Solution: The cartesian product of A and B is the set of all tuples (a, b) with $a \in A$ and $b \in B$. Thus, $A \times B = \{(0, 2), (0, 3), (1, 2), (1, 3)\}$.

Comments. Note that the elements of $A \times B$ are *tuples* (a, b) (where the order matters) rather than subsets $\{a, b\}$, so round parentheses, not braces, must be used.

Problem 2

(Multiple choice. Circle the correct answer.) For arbitrary sets A and B , the difference set $A - B$ is equal to

- (a) $\overline{B - A}$ (b) $\overline{A \cup B}$ (c) $\overline{B} \cup A$ (d) $\overline{B} \cap A$ (e) $\overline{A \cup B}$ (f) None of the above.

Solution: $A - B$ is the set of elements which are in A , but not in B , so it is equal to $\overline{B} \cap A$.

Also correct is $\overline{A \cup B}$. (A Venn diagram shows that the latter is the same as $\overline{B} \cap A$.)

Problem 3

(Multiple choice. Circle the correct answer.) How many functions are there from a 5-element set to a 7-element?

- (a) $5 \cdot 7$ (b) 5^7 (c) 7^5 (d) $7!/2!$ (e) $\binom{7}{5}$ (f) None of the above

Solution: For each of the 5 elements of the first set there are 7 ways assign the value of f at this element, so the total number of possible assignments is $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^5$. Thus, (c) is the correct answer.

Problem 4

Find a **proper** subset $A \subset \mathbb{N}$ and a **surjective** function f from A to \mathbb{N} , or prove that no such A and f exist.

Solution: A and f with these properties exist. For example, let $A = \{0, 2, 4, \dots\}$, i.e., the set of *even* integers in \mathbb{N} (obviously a proper subset of \mathbb{N}), and let f be defined by $f(n) = n/2$. Then f is a bijection (and thus, in particular, a surjection) from A to \mathbb{N} . Another example is obtained by taking $A = \{1, 2, 3, \dots\}$ (which again is a proper subset of \mathbb{N} since 0 is in \mathbb{N} , but not in A) and $f(n) = n - 1$ for $n = 1, 2, \dots$

Problem 5

1. Express the sum of the first 1000 **odd** positive integers, i.e., $1 + 3 + 5 + \dots + 1997 + 1999$, using summation notation, and then evaluate it. (Give a numerical value for this sum.)

Solution: The general form of a positive odd integer is $2i - 1$, with i a positive integer, so the given sum can be written as $\sum_{i=1}^{1000} (2i - 1)$. It is equal to

$$\sum_{i=1}^{1000} (2i - 1) = 2 \sum_{i=1}^{1000} i - \sum_{i=1}^{1000} 1 = 2 \frac{1000 \cdot 1001}{2} - 1000 = \boxed{10^6},$$

using the summation formula $\sum_{i=1}^n i = n(n + 1)/2$.

2. Find a simple formula for the product

$$P_n = \prod_{i=1}^n \prod_{j=i}^{i+1} j$$

for a general positive integer n .

Solution: The inner product is equal to $\prod_{j=i}^{i+1} j = i(i + 1)$, so

$$P_n = \prod_{i=1}^n (i(i + 1)) = (1 \cdot 2)(2 \cdot 3) \dots ((n - 1) \cdot n)(n \cdot (n + 1)).$$

On the right-hand side, each integer i with $2 \leq i \leq n$ appears exactly twice, and there is an additional factor of $n + 1$, appearing once. Thus the above simplifies to

$$\boxed{P_n = (n!)^2(n + 1)} \text{ or } \boxed{P_n = n!(n + 1)!}.$$

Problem 6

Determine the number of 5-letter words under the following restrictions:

1. with **at least one repeated letter**.

Solution: We use the complement trick: 26^5 is the total number of 5-letter words, $26!/21!$ is the number of such words with no repeated letters, so the number of such words with at least one repeated letter is $\boxed{26^5 - 26!/21!}$.

Comments. This is a standard birthday-type problem. There is no easy way to directly get the answer (i.e., without resorting to the complement trick).

2. with all letters **distinct and in alphabetical order**.

Solution: There are $\binom{26}{5}$ ways to pick the 5 letters for the word, and only one way to place these in alphabetical order, so the answer is $\boxed{\binom{26}{5}}$.

Comments. The above method is the only practical way to get the answer. Attempts to get the answer by counting the number of choices for the first letter, the second letter, etc., are bound to fail, because of the requirement that the letters have to be in alphabetical order. (The number of choices available for each position depends on how the earlier positions have been filled. For example, if the first letter is an A, there are 25 choices for the second letter, but if the first letter is a B, there are only 24, and if the first letter is a Z, there is none ...)

3. with **exactly** two distinct letters. (For example, AAFAFF counts, but not AAFAFX or AAAAA, since the latter two words contain three, respectively one, distinct letters.)

Solution: Proceed in two stages. First choose the two letters to be used; this can be done in $\binom{26}{2}$ ways. Next, fill each of the 5 positions with one of these two letters, keeping in mind that each letter has to appear at least once. The latter is equivalent to counting binary sequences of length 5 (say H/T sequences) in which each symbol occurs at least once. There are 2^5 such sequences total (namely, HHHHH, HHHHT, ..., TTTTH, TTTTT), but 2 of these consist of only one letter, leaving $2^5 - 2$ such sequences with both letters appearing. Multiplying the counts for the two stages, we get $\boxed{\binom{26}{2}(2^5 - 2)}$ as answer.

Problem 7

1. Determine the coefficient of x^9 in the expansion of $(2 - x^3)^{10}$.

Solution: By the binomial theorem,

$$(2 - x^3)^{10} = \sum_{k=0}^{10} \binom{10}{k} 2^k (-x^3)^{10-k}.$$

The term involving x^9 corresponds to the $k = 7$ term in the above expansion, so the coefficient of x^9 is $\boxed{\binom{10}{7} 2^7 (-1)}$.

2. Using the binomial theorem, prove that

$$\binom{100}{0} + \binom{100}{2} + \cdots + \binom{100}{98} + \binom{100}{100} = \binom{100}{1} + \binom{100}{3} + \cdots + \binom{100}{97} + \binom{100}{99}$$

Solution: The binomial theorem states

$$(x + y)^{100} = \sum_{k=0}^{100} \binom{100}{k} x^k y^{100-k}.$$

Applying this with $x = -1$ and $y = 1$, we get

$$0 = (1 - 1)^{100} = \sum_{k=0}^{100} \binom{100}{k} (-1)^k = \sum_{\substack{k=0 \\ k \text{ even}}}^{100} \binom{100}{k} - \sum_{\substack{k=0 \\ k \text{ odd}}}^{100} \binom{100}{k}.$$

Moving the latter sum to the other side gives

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{100} \binom{100}{k} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{100} \binom{100}{k},$$

which is the identity we set out to prove.

Comments. This was one of the illustrations of the binomial theorem worked out in class (and also in the text), with a general n in place of 100. The identity has a combinatorial interpretation, namely that for any finite set the number of subsets that have an even number of elements is the same as the number of subsets that have an odd number of elements. However, this interpretation would not be an acceptable “proof” of the identity (it’s nothing but a restatement of the identity in combinatorial language). Since the problem specifically asked to derive the identity *from the binomial theorem*, you had to proceed as above by starting from the general form of the binomial theorem, and then choosing particular values for x and y (namely, 1 and -1).

Problem 8

Determine the number of points (x, y, z) with integer coordinates in the first octant (i.e., with $x, y, z \geq 0$) for which the sum of all three coordinates is at most 13. (For example, $(2, 1, 3)$ or $(0, 3, 10)$ count, but $(1, 3, 10)$ or $(1, -2, 6)$ do not count.)

Solution: The problem amounts to counting solutions of

$$(*) \quad x + y + z \leq 13$$

with nonnegative integers. With an equality sign instead of an inequality sign, it would be a routine “donut counting” problem, and the answer would be $\binom{3+13-1}{13}$. The given version is one of the variations of the standard problem that was worked out in class. The trick is to introduce a new variable (like a “virtual” donut type), say w , and write $(*)$ as

$$(**) \quad x + y + z + w = 13$$

where x, y, z, w are any nonnegative integers. The latter equation $(**)$ has $\boxed{\binom{13+4-1}{13}}$ solutions.

Comments. Breaking the inequality $(*)$ down into cases $x+y+z = n$, for $n = 0, 1, \dots, 13$ and counting the number of solutions for each of these cases leads to a sum involving 14 binomial terms. This, however, does not qualify as a simple answer (see point 3 on the cover sheet), so only partial credit was given for an answer in this form. No credit was given for brute force attempts to count solutions by trying to list them all; because of the numbers involved, such attempts were doomed to fail.

Problem 9

Let $A = \{a_1, \dots, a_{10}\}$ be a set of 10 distinct positive two-digit integers (i.e., each a_i is at least 10 and at most 99).

For each subset $B \subseteq A$ let $f(B)$ denote the sum of all elements in B . (If $B = \emptyset$, set $f(B) = 0$.) Thus, f is a function from the power set $P(A)$ of A to \mathbb{N} . Prove that this function is **not** injective.

Solution: To show that the function is not injective, we need to show that there exist two distinct subsets at which f takes on the same value.

To do this, we apply the pigeonhole principle, with the possible values of f as pigeonholes, and the subsets of A as pigeons. Since A is a 10-element set, there are 2^{10} such “pigeons”.

On the other hand, since each element of A is a nonnegative integer ≤ 99 and A has 10 elements, the sum of the elements of any such subset must be an integer, between 0 and $10 \cdot 99$, so there are at most 991 possible values of f , i.e., we have *at most* 991 pigeonholes.

Since $2^{10} = 1024 > 991$, we have more pigeons than pigeonholes, so some pigeonhole must hold more than one pigeons. By our definition of pigeons and pigeonholes, this means that there exist two (distinct) subsets $B_1, B_2 \subseteq A$ with $f(B_1) = f(B_2)$. This is what we wanted to show.

Problem 10

Let

$$f(n) = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

Guess a simple formula for $f(n)$, and then use induction to prove that this formula holds for all positive integers n .

Solution: The values of f for $n = 1, 2, 3$ are $1/3, 2/5, 3/7$, suggesting the formula

$$(*) \quad f(n) = \frac{n}{2n+1}.$$

We prove this by induction.

Base step: For $n = 1$, we have $f(1) = 1/(1 \cdot 3) = 1/3$, so $(*)$ holds in this case.

Inductive step: Assume $(*)$ holds for $n = k$, where k is a positive integer. Then

$$\begin{aligned} f(k+1) &= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\ &= f(k) + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \quad (\text{by def. of } f(k)) \\ &= \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \quad (\text{by ind. hyp.}) \\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{(k+1)}{2(k+1)+1}. \end{aligned}$$

Thus, $(*)$ holds for $n = k + 1$, and the induction is complete.