

- **Grading:** The exam was worth 100 points. The breakdown per problem is given on the coversheet. For multipart problems, each part was worth 5 points.
- **Exam statistics:** Scores ranged from 39 to 100. The average was 71, and the median score was 73.
- **Access to online scores:** See the course webpage for a link to the online score system, an explanation of the score display, and an approximate letter grade correspondence for your total score.
- **Solutions:** Solutions, along with some remarks about common errors, follow below. Check these solutions first before asking questions about the grading.

## Exam Solutions

1. Let  $A = \{1, 2, 3\}$ ,  $B = \{3, 4\}$ , and  $C = \{4\}$ . For the first four parts, write down the requested set explicitly *using proper set-theoretic notation*. The final part is a true/false question, involving the set  $A$  above. (For this problem, answers are sufficient. Just write down the sets requested; no justification needed.)

(a)  $(A - B) \cup (B - A) = \boxed{\{1, 2, 4\}}$

(b)  $A - (B - C) = \boxed{\{1, 2\}}$

(c)  $B \times C = \boxed{\{(3, 4), (4, 4)\}}$

- (d) (True/false) With the set  $A = \{1, 2, 3\}$  defined as above, determine which of the following statements about the power set  $P(A)$  are true, and which are false. Mark those that are true by  $\boxed{\text{T}}$ , and those that are false by  $\boxed{\text{F}}$ . If you are unsure, leave the answer blank. (There will a small penalty for an incorrectly marked answer.)

- A.  $\{2, 3\} \in P(A)$   
 B.  $\{2, 3\} \subseteq P(A)$   
 C.  $\emptyset \in P(A)$   
 D.  $\emptyset \subseteq P(A)$

**Solution:**  $\boxed{\text{A,C,D are true, B is false.}}$  Reason: By definition,  $P(A)$  is the set whose elements are the subsets of  $\{1, 2, 3\}$ . Since  $\{2, 3\}$  and  $\emptyset$  are among these subsets, they are elements of  $P(A)$ , so A and C are true. D is true since any set contains the empty set **as a subset**. However, B is false, since  $\{2, 3\}$  is an **element**, not a subset of  $P(A)$ .

**Grading note:** 5 points if all 4 answers are correct. 4 points for 3 right and 1 wrong answer, 2 points for 2 right and 2 wrong answers, and 0 points for 3 or more incorrect answers.

2. Let  $f$  and  $g$  be functions from  $\mathbb{Z}$  to  $\mathbb{Z}$  defined by

$$f(n) = n^3 + 1, \quad g(n) = \lfloor n/2 \rfloor,$$

where  $\lfloor \dots \rfloor$  denotes the floor function.

- (a) Is  $f$  an **onto** function from  $\mathbb{Z}$  to  $\mathbb{Z}$ ? If not, give a specific counterexample. (If yes, no further justification is needed.)

**Solution:**  $\boxed{\text{NO}}$  The number 3 is not of the form  $n^3 + 1$  with  $n \in \mathbb{Z}$ , and hence not in the range of  $f$ . (This is a variation on Problem 2.3:12/13 from HW 1.)

- (b) Is  $f$  a **one-to-one** function from  $\mathbb{Z}$  to  $\mathbb{Z}$ ? If not, give a specific counterexample. (If yes, no further justification is needed.)

**Solution:** YES (See Problem 2.3:12 from HW 1.)

- (c) Is  $g$  a **one-to-one** function from  $\mathbb{Z}$  to  $\mathbb{Z}$ ? If not, give a specific counterexample. (If yes, no further justification is needed.)

**Solution:** NO Since  $\lfloor 1/2 \rfloor = 0$  and  $\lfloor 0/2 \rfloor = 0$ ,  $g(1)$  and  $g(0)$  are both 0.

- (d) Is there a set  $A \subseteq \mathbb{Z}$  such that  $g$  becomes a bijection from  $A$  to  $\mathbb{Z}$ . If yes, specify the set  $A$ . (If no, no further justification is needed.)

**Solution:** YES Take  $A$  to be the set of *even* integers.

3. Evaluate the following sums and products. (Here  $n$  denotes a (general) positive integer. In the last part,  $p$  is a real number with  $0 < p < 1$ .)

(a)  $\prod_{i=0}^n (-2)$   $(-2)^{n+1}$

(b)  $\prod_{i=1}^n i^n$   $1^n 2^n \dots n^n = n!^n$

(c)  $\sum_{i=1}^n \sum_{j=1}^i 1$   $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

(d)  $\sum_{i=0}^n p^i (1-p)^{n-i}$

**Solution:** This sum is of the same type as 9(h) from HW 2 (with  $p$  and  $1-p$  in place of  $x$  and  $y$ ). To evaluate it, factor out  $(1-p)^n$  and convert the sum to a geometric series with ratio  $r = p/(1-p)$ :

$$(1-p)^n \sum_{i=0}^n \left( \frac{p}{1-p} \right)^i = (1-p)^n \frac{\left( \frac{p}{1-p} \right)^{n+1} - 1}{\frac{p}{1-p} - 1}$$

4. State the official (formal) mathematical definitions for the given terms, using proper mathematical language and notation.

- (a) The **graph** of a function  $f : A \rightarrow B$  is defined as the set ...

$$\{(a, b) \mid a \in A, b = f(a)\} \text{ (p. 142 in the text)}$$

- (b) Two sets  $A$  and  $B$  are said to have the **same cardinality** if ...

$$\text{there exists a bijection from } A \text{ to } B \text{ (Def. 4, p. 158)}$$

**Remark:** Saying that  $A$  and  $B$  have the same cardinality if “they have the same number of elements” would not make sense for infinite sets. The only proper way to define cardinality is via bijections.

- (c) A set  $A$  is called **countable** if ...

$$\text{there exists a bijection from } \mathbb{Z}_+ \text{ to } A \text{ (Def. 5, p. 158)}$$

Equivalently,  $A$  is countable if it *has the same cardinality as*  $\mathbb{Z}_+$ .

5. (Multiple choice. Circle the correct answer.) Suppose we want to prove a statement  $P(n)$  by induction.

- (a) Which of the following is the **induction hypothesis** in a standard induction proof? Circle the correct answer.

- A. “ $P(n)$  is true for all  $n \in \mathbb{Z}_+$ ”  
 B. “ $P(n)$  is true for  $n = 1$ .”  
 C. “ $P(n)$  is true for  $n = 1, 2, \dots, k$ .”  
 D. “ $P(n)$  is true for  $n = k$  and  $n = k - 1$ .”  
 E. “ $P(n)$  is true for  $n = k$ .”  
 F. “ $P(n)$  is true for  $n = k + 1$ .”

E: “ $P(n)$  is true for  $n = k$ .”

- (b) Which of the following is the **induction hypothesis** in a proof by **strong induction**? Circle the correct answer.

- A. “ $P(n)$  is true for all  $n \in \mathbb{Z}_+$ ”  
 B. “ $P(n)$  is true for  $n = 1$ .”  
 C. “ $P(n)$  is true for  $n = 1, 2, \dots, k$ .”  
 D. “ $P(n)$  is true for  $n = k$  and  $n = k - 1$ .”  
 E. “ $P(n)$  is true for  $n = k$ .”  
 F. “ $P(n)$  is true for  $n = k + 1$ .”

C: “ $P(n)$  is true for  $n = 1, 2, \dots, k$ .”

6. Let  $a_1 = 1$ ,  $a_2 = 2$ , and  $a_n = a_{n-1} + a_{n-2}$  for  $n \geq 3$ . Prove that for all  $n \in \mathbb{Z}_+$ ,

$$a_n \geq \left(\frac{3}{2}\right)^{n-1}.$$

Your write-up must include all steps, in the correct logical order, and with appropriate justifications for each step (e.g., “by ind. hyp. applied to ...”, “by equation (1)”).

**Solution:** We use strong induction to prove that, for all  $n \in \mathbb{Z}_+$ ,

$$(*) \quad a_n \geq \left(\frac{3}{2}\right)^{n-1}.$$

**Base step:** Our base consists of the first two cases,  $n = 1$  and  $n = 2$ . When  $n = 1$ , the left side of  $(*)$  is  $a_1 = 1$ , and the right side is  $(3/2)^0 = 1$ , so  $(*)$  holds for  $n = 1$ . When  $n = 2$ , the left side of  $(*)$  is  $a_2 = 2$ , and the right side is  $(3/2)^1 = 3/2$ , so the inequality  $(*)$  holds also for  $n = 2$ .

**Induction step:** Let  $k \geq 2$  be given and suppose  $(*)$  is true for all  $n = 1, 2, \dots, k$ . Then

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \quad (\text{by recurrence for } a_n) \\ &\geq \left(\frac{3}{2}\right)^{k-1} + \left(\frac{3}{2}\right)^{(k-1)-1} \quad (\text{by strong induction hypothesis with } n = k \text{ and } n = k - 1) \\ &= \left(\frac{3}{2}\right)^k \left( \left(\frac{3}{2}\right)^{-1} + \left(\frac{3}{2}\right)^{-2} \right) \quad (\text{by algebra (factor out } (3/2)^k)) \\ &= \left(\frac{3}{2}\right)^k \left( \frac{2}{3} + \frac{4}{9} \right) \quad (\text{more algebra}) \\ &= \left(\frac{3}{2}\right)^k \frac{10}{9} \quad (\text{final piece of algebra}) \\ &> \left(\frac{3}{2}\right)^k = \left(\frac{3}{2}\right)^{(k+1)-1}. \end{aligned}$$

Thus, (\*) holds for  $n = k + 1$ , and the proof of the induction step is complete.

**Conclusion:** By the strong induction principle, it follows that (\*) is true for all  $n \in \mathbb{Z}_+$ .

**Remarks:** Except for a shift in indexing ( $n \rightarrow n + 1$ , which turns the sequence  $a_n$  into the Fibonacci sequence), this problem was worked out in class.

Here are some common errors:

- (1) **The induction must start with  $n = 1$  in the base step, not  $n = 3$ .** This is because we need to prove the asserted inequality for all  $n \in \mathbb{Z}_+$ . With  $n = 3$  as base case, the inequality would only be proved for  $n \geq 3$ .
- (2) **The base step must include a proof for the first two cases,  $n = 1$  and  $n = 2$  (and not just  $n = 1$ ).** This is because the induction step requires the inequality to hold for the **two** preceding cases ( $n = k$  and  $n = k - 1$ ). With only  $n = 1$  as base case, the induction step would break down for  $k = 1$  since in that case  $k - 1 = 0$ , which is out of our range  $\mathbb{Z}_+$ .
- (3) **The most crucial part of the argument is the place where the induction hypothesis is being used (second line above) and the subsequent algebraic calculations (between the second and fifth line).** These steps have to be present in full detail, they have to be correct (in terms of the algebra performed), and they must include appropriate justifications (“by induction hypothesis”, etc.).