

## Grading notes

- **Exam statistics:** The maximal score was 100 points; the median score was 70.
- **Score breakdown:** The breakdown per problem is as follows: Problem 1: 30 pts (5 each); Problem 2: 15 pts (7+8); Problem 3: 15 pts (7+8); Problem 4: 20 pts (6+6+8); Problem 5: 10 pts. Problem 6: 10 pts.
- **Access to online scores:** See the course webpage for a link to the online score system, an explanation of the score display, and an approximate letter grade correspondence for your total score.
- **Solutions:** Solutions, along with some remarks about common errors, follow below. Check these solutions first before asking questions about the grading.

## Exam Solutions

1. **Quickies.** For the questions below, just provide the requested answer—no explanation or justification required.

As always, answers should be left in raw, unevaluated form, involving binomials, factorials, powers, etc. (such as  $\binom{10}{3}2^3$ ). Do **not** use notations  $C(n, k)$  or  $P(n, k)$  in your answers.

All of the questions have a simple answer and require virtually no hand computations. Write legibly and circle or box your answer.

- (a) What is the number of 10-element subsets of a 100 element set?  $\binom{100}{10}$
- (b) What is the number of functions from the set  $\{1, 2, \dots, 100\}$  to the set  $\{0, 1\}$ ?  $2^{100}$
- (c) What is the number of 7-letter upper-case words that contain the letter A?  $26^7 - 25^7$
- (d) What is the number of 7-letter upper-case words, whose letters are *distinct and occur in alphabetically increasing order*. (E.g., BDFKSTX is counted, but not BFDKSTX (since F and D are not in alphabetical order), nor BDDKSTX (since D is repeated).)  
**Solution:** This was worked out in class. There are  $\binom{26}{7}$  ways to choose the 7 letters, and exactly one way to place them in alphabetical order, so the total count is  $\binom{26}{7}$
- (e) What is the probability that in 10 rolls of an ordinary die exactly 4 sixes come up?  $\binom{10}{4}(1/6)^4(5/6)^6$
- (f) Given that  $P(A) = 0.5$ ,  $P(B) = 0.8$ , and  $A \subseteq B$ , what is the conditional probability  $P(A|B)$ ?  
**Solution:**  $P(A|B) = P(A \cap B)/P(B) = P(A)/P(B) =$  $0.5/0.8$

2. (a) Evaluate the sum

$$\binom{100}{0} + 2\binom{100}{1} + 2^2\binom{100}{2} + \dots + 2^{99}\binom{100}{99} + 2^{100}\binom{100}{100}.$$

**Solution:** By the Binomial Theorem, the given sum is

$$\sum_{k=0}^{100} 2^k \binom{100}{k} = (2 + 1)^{100} = \span style="border: 1px solid black; padding: 2px;"> $3^{100}$$$

- (b) Find the coefficient of  $x^{10}$  in  $(2x + \frac{1}{x})^{100}$ .

**Solution:** (This is similar to Problem 5.4:10 from HW 4.) By the binomial theorem

$$\left(x + \frac{1}{x}\right)^{100} = \sum_{k=0}^{100} (2x)^k \left(\frac{1}{x}\right)^{100-k} \binom{100}{k} = \sum_{k=0}^{100} 2^k x^{2k-100} \binom{100}{k}.$$

The  $x^{10}$  term corresponds to  $k = 55$ , and its coefficient is  $\boxed{2^{55} \binom{100}{55}}$ .

3. Determine the number of ways to arrange the 11 letters of the word MISSISSIPPI under the given conditions. (Note that this word has 4 S's, 4 I's, 2 P's, and 1 M.)

- (a) If all 4 I's must be adjacent (as in MSPPIIIIPSMSS).

**Solution:** This is a standard word counting problem that came up in HW 5. The trick is to treat the I-block as a single unit  $\boxed{\text{IIII}}$ . We then have to count permutations of 4 S's, 2 P's, 1 M, and 1  $\boxed{\text{IIII}}$ , a total of 8 symbols. The number of these is

$$\boxed{\frac{8!}{4!2!1!1!}}$$

(The numerator,  $8!$ , counts the number of permutations of 8 *distinct* objects, and the factors  $4!$ , and  $2!$  in the denominator compensate for the repetition of the letters  $S$  and  $P$ .)

**Remark:** An alternative way to arrive at the correct answer is by filling 8 slots with 4 S's, 2 P's, 1 M, and 1  $\boxed{\text{IIII}}$ , one at a time: There are  $\binom{8}{4}$  ways to pick slots for the 4 letters  $S$ ,  $\binom{4}{2}$  ways to pick slots for the 2 letters  $P$ ,  $\binom{2}{1}$  ways to pick slots for the letter  $M$ , and  $\binom{1}{1}$  ways to pick the remaining slot for the letter  $\boxed{\text{IIII}}$ . This gives a total count of  $\binom{8}{4} \binom{4}{2} \binom{2}{1} \binom{1}{1}$ . Expressing the binomial coefficients in terms of factorials and canceling shows that this is the same answer as the one obtained above.

- (b) If the letters  $I$  are **not** allowed be adjacent. (For example, MISSIISSPPI would not be counted, since it has two adjacent I's).

**Solution:** This is of the same type as the "win streak" problem discussed in class, or the "men/women in a line" problem from HW 4. There are 4 I's and 7 other letters. Considering the other letters as separators ("bars"), we have 8 spaces created by these separators in which to place the I's. There are  $\binom{8}{4}$  ways to choose 4 spaces out of these 8 for the 4 I's. Once the I's have been placed, we have 7 slots left to fill with the remaining 7 letters. We can do this in  $\frac{7!}{4!2!1!}$  ways, by the same argument as above. Hence, the total count is

$$\boxed{\binom{8}{4} \frac{7!}{4!2!1!}}$$

4. The following are poker problems. Recall that a card deck consists of 52 cards, split into four equal *suits* (spades, etc.), of 13 cards each. There are 13 possible *kinds* of cards (ace, etc.), and a suit contains exactly one of each kind.

The following questions concern probabilities for 7-card *poker hands*, i.e., unordered selections of 7 out of 52 cards.

- (a) What is the probability that **exactly** 2 of the 7 cards are diamonds?

**Solution:** There are  $\binom{13}{2}$  ways to choose the 2 diamond cards, and  $\binom{39}{5}$  ways to choose 5 cards from the 39 non-diamond cards on the deck. Thus, the probability is

$$\boxed{\frac{\binom{13}{2} \binom{39}{5}}{\binom{52}{7}}}.$$

- (b) What is the probability that the 7 cards are all of different kinds?

**Solution:** There are  $\binom{13}{7}$  ways to choose 7 different kinds, and for each of these 7 kinds there are 4 ways to choose a suit. Thus, the total number of poker hands with 7 different kinds is  $\binom{13}{7}4^7$ , and its probability is

$$\frac{\binom{13}{7}4^7}{\binom{52}{7}}.$$

- (c) What is the probability that 4 of the cards are from one suit, and 3 are from another suit?

**Solution:** This is similar to the “full house” probability in standard poker, worked out in class. We proceed in stages: First pick a suit for the *quadruple* of cards (4 choices). Then pick 4 cards from this suit ( $\binom{13}{4}$  choices). Next pick a suit for the *triple* of cards (3 choices, since it cannot be the suit chosen for the quadruple). Finally, pick 3 cards from that suit ( $\binom{13}{3}$  choices). Multiplying out these counts and dividing by  $\binom{52}{7}$  gives the requested probability:

$$\frac{4\binom{13}{4}3\binom{13}{3}}{\binom{52}{7}}.$$

**Remark:** The number of choices for the suits is  $4 \cdot 3$ , not  $\binom{4}{2} (= 4 \cdot 3/2)$ . This is a rather subtle, but important, point. To see this note that the  $\binom{4}{2}$  formula counts unordered pairs of suits and thus would not distinguish between, for example, (A) a quadruple of diamonds and a triple of spades, and (B) a triple of spades and a quadruple of diamonds, but clearly (A) and (B) represent different hands. (However, if we had to pick suits for two triples, the situation would be different: a triple of spades and a triple of diamonds is indistinguishable from a triple of diamonds and a triple of spades, so the appropriate count here would be  $\binom{4}{2}$ ).

5. Suppose there are 4 varieties of donuts: Chocolate (C), Glazed (G), Pumpkin (P), and Raspberry (R). Find the number of ways one can select *at most* 30 donuts such that the selection includes at least 1 C donut, 2 G donuts, 3 P donuts, and 4 R donuts.

**Solution:** First, to satisfy the above requirements we preselect the minimum number of each type: 1 C, 2 G, 3 P, 4 R. We then have *at most*  $30 - 1 - 2 - 3 - 4 = 20$  donuts left to choose, now without any restrictions.

With *exactly* 20 instead of *at most* 20, this count would be  $\binom{20+4-1}{20}$ , by the standard donut counting argument. To account for the “at most” condition, we imagine having a fifth type of donut, V (“virtual”), and consider the difference between the maximal number of donuts we can choose (i.e., 20) and the number of donuts we actually choose to be donuts of this virtual type. This gives a count of

$$\binom{20+5-1}{20},$$

by the standard donut type counting argument applied with 5 types instead of 4.

**Remarks:** The two crucial ingredients in this proof are: (i) The “preselection” trick to satisfy the given minimum requirements. This reduces the number of donuts to work with to 20. (ii) The “virtual donut” (or extra variable) trick to account for the “at most” condition. This increases the number of types to 5. Without this trick, one would get  $\binom{20+4-1}{20}$ , which would count the number of selections of *exactly* 30 donuts (with the same minimum requirements).

6. **Using the pigeonhole principle**, show that any set of 76 positive integers  $\leq 100$  must contain 4 consecutive integers.

Your solution must use (a suitable version of) the pigeonhole principle, and must include *precise* definitions of (i) the “pigeonholes” and (ii) the “pigeons”, along with the counts (“ $N$ ”) of pigeons and (“ $k$ ”) of pigeonholes.

**Solution:** This is a variation on the “consecutive house number” problem, 5.2:14, from hw 3. Define pigeons and pigeonholes as follows:

- **Pigeons:** The numbers in the given set. **Count:** There are 76 numbers in the set, so  $N = 76$ .
- **Pigeonholes:** The sets

$$(1) \quad H_1 = \{1, 2, 3, 4\}, H_2 = \{5, 6, 7, 8\}, \dots, H_{25} = \{97, 98, 99, 100\}.$$

**Count:** There are 25 such sets, so the number of pigeonholes is  $k = 25$ .

Since  $\lceil N/k \rceil = \lceil 76/25 \rceil = 4$ , the generalized pigeonhole principle guarantees that there exists a “pigeonhole” that contains 4 “pigeons”. By the above definition of pigeons and pigeonholes this means one of the sets  $H_1, \dots, H_{25}$  in (1) contains 4 numbers from the given set, and those 4 numbers must be consecutive, by the definition of the sets  $H_i$ .

**Remarks:** (i) In the above application of the pigeonhole principle it is crucial that the pigeonholes be defined precisely as in (1). In particular, the sets must be nonoverlapping in order for the counts to work out. If one were to define the sets as

$$(1') \quad \{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4, 5, 6\}, \dots$$

one would end up with a total of 97 sets, and hence 97 pigeonholes. But this is more than the 76 pigeons, so the pigeonhole principle wouldn’t guarantee that some hole contains 4 pigeons.

(ii) The number 76 in the problem is best-possible; the statement would be false if 76 were replaced by 75. (An example is the set  $\{1, 2, 3, 5, 6, 7, 9, 10, 11, \dots, 95, 97, 98, 99\}$  which has 75 elements but no 4 consecutive integers.) In the above argument, the crucial inequality is  $76/25 > 3$ . This ensures that some hole has 4 pigeons. With 75 instead of 76, one would have  $\lceil 75/25 \rceil = 3$  and the pigeonhole principle would only guarantee that some hole has 3 pigeons, which is not enough.

An easy way to convince yourself that an alternate argument cannot *possibly* be right is to check if it would give the same conclusion when 76 is replaced by 75 (in which case we know the result cannot be true). For example, if the pigeonhole application involves  $\binom{76}{4}$  pigeons and 75 pigeonholes, then replacing 76 by 75 would not change the relative numbers of pigeons and pigeonholes.