

### Problem 1

Suppose you roll 4 ordinary dice. Determine the following probabilities:

1. The probability that **at least** one 6 comes up.

**Solution:** By the complement trick, this is  $1 - P(\text{no 6 in 4 rolls})$ , i.e.,  $\boxed{1 - (5/6)^4}$ .

2. The probability that **exactly** one 6 comes up.

**Solution:** This is given by the binomial distribution:  $\boxed{\binom{4}{1}(1/6)^1(1 - 1/6)^3 = \frac{4 \cdot 5^3}{6^4}}$ .

3. The probability that all four numbers showing up are distinct.

**Solution:** This is a birthday-type probability:  $\boxed{\frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5}{18}}$ .

4. The probability that all four dice show the same number.

**Solution:** There are 6 ways to choose the number showing up on the dice, and for a given number, the probability that all four dice show this particular number is  $1/6^4$ .

Thus, the probability in question is  $6 \cdot (1/6^4) = \boxed{1/6^3}$ .

### Problem 2

Consider the recurrence  $a_n = -2na_{n-1}$ .

1. (Multiple choice. Circle the correct answer.) Determine whether this recurrence is
  - (a) Linear and homogeneous
  - (b) Linear and nonhomogeneous
  - (c) Nonlinear and homogeneous
  - (d) Nonlinear and nonhomogeneous

**Solution:** (a) The recurrence is linear and homogeneous. (Note that in a linear recurrence the coefficients of the terms  $a_n$ ,  $a_{n-1}$ , etc., don't have to be constant and can be arbitrary functions of  $n$ .)

2. Solve this recurrence with initial condition  $a_0 = 1$ .

**Solution:** By iteration,

$$a_n = -2na_{n-1} = (-2)^2n(n-1)a_{n-2} = \cdots = (-2)^n n(n-1) \cdots 2 \cdot 1 a_0 = \boxed{(-2)^n n!}.$$

### Problem 3

Let  $a_n$  denote the number of ways to climb  $n$  stairs if stairs can be climbed 2 or 3 at a time. Set up a recurrence for  $a_n$ ; be sure to clearly explain how each term in this relation arises.

**Solution:** Consider the last step when climbing  $n$  stairs. If this is a 2-stair step, your position before taking this step is at the end of  $n - 2$  stairs, and there are  $a_{n-2}$  ways to reach this position. If it is a 3-stair step, your position before taking the step is at the end of  $n - 3$  stairs, and there are  $a_{n-3}$  ways to reach this position. Since these are the only possibilities for the last step, the sum of these two counts gives the total number of ways to climb  $n$  stairs, i.e., we have the recurrence  $\boxed{a_n = a_{n-2} + a_{n-3}}$ .

**Problem 4**

1. Find the **general** solution to the recurrence  $a_n = 6a_{n-1} - 9a_{n-2}$ .

**Solution:** The associated characteristic equation is  $r^2 - 6r + 9 = 0$ , which has a double root at  $r = 3$ . Thus, the general solution is of the form  $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ .

2. Find a **particular** solution to the recurrence  $a_n = 6a_{n-1} - 9a_{n-2} + 28n$ .

**Solution:** The nonhomogeneous term is  $28n$ , so we seek a solution of the form  $(*)$   $a_n = An + B$  with constants  $A$  and  $B$ . Substituting  $(*)$  into the recurrence, we get

$$An + B = 6(A(n-1) + B) - 9(A(n-2) + B) + 28n = n(-3A + 28) + (12A - 3B).$$

Equating coefficients of  $n$  and of 1 (i.e., the constant terms) on both sides, we get the equations  $A = -3A + 28$  and  $B = 12A - 3B$ , so  $A = 7$  and  $B = 3A = 21$ . Thus, a particular solution is  $a_n = 7n + 21$ .

**Problem 5**

1. Write down the generating function  $G(x)$  for the number  $a_r$  of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = r,$$

where  $x_1, x_2, x_3$  are arbitrary nonnegative integers, and  $x_4$  is a nonnegative multiple of 10 (i.e., the possible values of  $x_4$  are  $0, 10, 20, \dots$ ). Simplify your formula as much as possible, e.g., by evaluating power series in closed form.

**Solution:** There are four factors contributing to  $G(x)$ , one for each of the four variables. The factors corresponding to  $x_1, x_2, x_3$  are  $(1 + x + x^2 + \dots) = (1 - x)^{-1}$  each, and the factor corresponding to  $x_4$  is  $1 + x^{10} + x^{20} + \dots = (1 - x^{10})^{-1}$ . Thus,  $G(x) = (1 - x)^{-3}(1 - x^{10})^{-1}$ .

2. **Using the generating function obtained above**, compute  $a_{15}$ , the number of solutions to the above equation with  $r = 15$ . (Give a numerical answer, e.g., 571. You must derive this answer via generating functions. No credit will be given for brute force attempts, or attempts using combinatorial arguments of the type that came up in Chapter 4.)

**Solution:** Expanding the factors  $(1 - x)^{-3}$  and  $(1 - x^{10})^{-1}$  gives

$$G(x) = \left( \sum_{k=0}^{\infty} \binom{3+k-1}{k} x^k \right) (1 + x^{10} + x^{20} + \dots).$$

$a_{15}$  is the coefficient of  $x^{15}$  in the above series, i.e.,

$$a_{15} = \binom{3+15-1}{15} + \binom{3+5-1}{5} = \binom{17}{15} + \binom{7}{5} = \frac{17 \cdot 16}{2} + \frac{7 \cdot 6}{2} = 136 + 21 = \boxed{157}.$$

**Problem 6**

Find the coefficient of  $x^{10}$  in the power series expansions of the following functions. (Your answer can be left in "raw" form.)

1.  $\sqrt{1+2x}$

**Solution:** By the extended binomial theorem,

$$\sqrt{1+2x} = (1+2x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (2x)^k,$$

so the coefficient of  $x^{10}$  is  $\boxed{2^{10} \binom{1/2}{10}}$ .

2.  $\exp(-x^2)$

**Solution:** By the exponential series,

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!},$$

so coefficient of  $x^{10}$  is  $(-1)^5/5! = \boxed{-1/5!}$ .

3.  $\frac{x}{2+5x}$

**Solution:** By the geometric series expansion,

$$\frac{x}{2+5x} = \frac{x}{2} \cdot \frac{1}{1 - (-5/2)x} = \frac{x}{2} \sum_{k=0}^{\infty} ((-5/2)x)^k,$$

so coefficient of  $x^{10}$  is  $(1/2)(-5/2)^9 = \boxed{-5^9 2^{-10}}$ .