

**Problem 1**

Consider 101 sets with the following properties: (i) each set has 1000 elements; (ii) the intersection of any two distinct sets has 20 elements; (iii) the intersection of any three pairwise distinct sets has 6 elements; (iv) the intersection of four or more pairwise distinct sets is empty. How many elements are there in the union of the union of these 101 sets? Give a numerical answer. (The calculations are not involved and the answer turns out to have a surprisingly simple form. (You'll know it when you see it.) Brute force attempts won't work here, so don't even try!)

**Solution:** Denoting the sets by  $A_1, \dots, A_{101}$ , we are given that  $|A_i| = 1000$ ,  $|A_i \cap A_j| = 20$ ,  $|A_i \cap A_j \cap A_k| = 6$ , and  $|A_i \cap A_j \cap A_k \cap A_l| = 0$  for any pairwise distinct  $i, j, k, l$ . We need to find,  $|\cup_{i=1}^{101} A_i|$ . By inclusion/exclusion, this is

$$\begin{aligned} \left| \bigcup_{i=1}^{101} A_i \right| &= \sum_{i=1}^{101} |A_i| - \sum_{1 \leq i < j \leq 101} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 101} |A_i \cap A_j \cap A_k| \\ &= 101 \cdot 1000 - \binom{101}{2} \cdot 20 + \binom{101}{3} \cdot 6 \\ &= 101 \cdot 1000 - \frac{101 \cdot 100}{2!} \cdot 20 + \frac{101 \cdot 100 \cdot 99}{3!} \cdot 6 \\ &= 101 \cdot 100 \cdot 99 = \boxed{999900} \end{aligned}$$

**Problem 2**

1. Write down an **exact** formula for  $D_n$ , the number of derangements of  $n$  objects.

**Solution:**

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

2. Write down an **approximate** value for  $D_n$  (involving a famous constant) when  $n$  is large.

**Solution:**  $D_n \approx n!e^{-1}$ .

3. How many ways are there to list the numbers  $1, 2, \dots, 20$  in some order such that exactly 5 of these numbers are in their "natural" positions? For example, 7 is in its natural position if it is the 7-th number in the list. Your answer can be left in raw form, and may involve factorials, binomial coefficients, as well as the function  $D_n$ , but no summations.

**Solution:** There are  $\binom{20}{5}$  ways to pick the numbers that are to remain in their natural positions, and  $D_{15}$  ways to derange the remaining 15 numbers, giving a

total of  $\boxed{\binom{20}{5} D_{15}}$  ways for an arrangement of the requested type.

**Problem 3**

Consider the relation  $R$  on the set  $\{1, 2, 3, 4\}$  defined by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

(That is,  $(i, j) \in R$  if the  $ij$ -th entry in this matrix is 1, and  $(i, j) \notin R$  if the  $ij$ -th entry is 0.) Is this relation an equivalence relation on the set  $\{1, 2, 3, 4\}$ ? If it is, explain briefly why it has each of the relevant properties. If not, state which of the properties of an equivalence relation it fails, and for each of these properties give a concrete example for which the property does not hold.

**Solution:** The properties required of an equivalence relation are (i) reflexivity, (ii) symmetry, and (iii) transitivity. For each of these we check whether it holds:

- (i) The relation is **not reflexive** since the diagonal entries are all 0; for example,  $(1, 1) \notin R$ .
- (ii) The relation is **symmetric** since the matrix is symmetric.
- (iii) The relation is **not transitive**, since, for example,  $(2, 1) \in R$ ,  $(1, 4) \in R$ , but  $(2, 4) \notin R$ . (Examples such as these can be easily discovered by drawing the graph representing the relation.)

**Problem 4**

Consider the graph  $K_{10}$ , the complete graph with 10 vertices.

1. How many edges does this graph have? (Hint: Don't try to draw the graph and count!)

**Solution:** There is one edge corresponding to each unordered pair of vertices, so the number is  $\binom{10}{2} = 45$ . (Alternatively, one can argue via the handshake theorem: There are 10 vertices, each having degree 9, so the sum of all degrees is 90. By the handshake theorem, this is twice the number of edges, so there are  $90/2 = 45$  edges.)

2. Does this graph have an Euler circuit? Explain!

**Solution:** No. An Euler circuit exists only if the graph is connected and each vertex has even degree. In the given graph, each vertex has degree 9 (as there are 9 edges connecting this vertex to the other 9 vertices), so, in fact, *none* of the vertices has even degree.

**Problem 5**

Consider the following pairs of graphs. In each case, determine whether the graphs are isomorphic. If they are, give a one-to-one matchup of vertices. If they are not, explain clearly why they cannot be isomorphic.

1. **Solution:** The first pair of graphs is **not isomorphic**, as can be seen by comparing the degrees of the vertices in the two graphs: The first graph has three vertices of degree 2, but the second only two.
2. **Solution:** The second pair of graphs is **isomorphic**, a possible match-up being  $a - 1, b - 3, c - 5, d - 7, e - 2, f - 4, g - 6$ .

### Problem 6

(Last, but not least, in honor of NFL draft day, a problem on NFL scheduling.) Suppose the NFL commissioner decides to expand the league to 17 teams per conference, so that there are a total of 34 teams in the NFL, 17 in the AFC and 17 in the NFC. Suppose also that the commissioner desires a schedule that has each team playing exactly 15 games per season, all against different opponents *within the same conference*. Is such a schedule possible? Explain clearly, citing appropriate theorems.

**Solution:** The desired schedule is impossible. To see this, focus on a *single conference*, and model the games within this conference by a graph with the 17 teams as vertices and edges corresponding to games on the schedule. The key now is to apply the **Handshake Theorem** to this graph: Since each team is supposed to play 15 games, each vertex has degree 15. Since there are 17 vertices, the sum of all degrees is  $15 \cdot 17 = 255$ , and thus an odd number. However, by the Handshake Theorem the sum of degrees in an undirected multigraph is always even (and equal to twice the number of edges), so this is impossible.

**Remark:** The given numbers, 17 teams per conference and 15 games per team, are critical in order for the above argument to work, and the argument breaks down if one of these numbers is replaced by an even number. For example, with 14 games played per team instead of 15, the sum of degrees would be an even number, and no contradiction to the handshake theorem would arise. The same is true if there are 16 instead of 17 teams per conference, each playing 15. Indeed, in both of these cases, a valid schedule can be easily constructed.