

Grading notes

- **Exam statistics:** The median score was 70, and the average 69. This is similar to the score statistics on the earlier midterms.
- **Score breakdown:** The breakdown per problem is as follows: Problem 1: 30 pts (5 each); Problem 2: 20 pts (5 each); Problem 3: 10 pts (5+5); Problem 4: 10 pts (5+5); Problem 5: 15 pts (5 each). Problem 6: 15 pts (5+10).
- **Access to online scores:** See the course webpage for a link to the online score system, an explanation of the score display, and an approximate letter grade correspondence for your total score.
- **Solutions:** Solutions, along with some remarks about common errors, follow below. Check these solutions first before asking questions about the grading.

Exam Solutions

1. **Quickies:** For the questions below, just provide the requested answer—no explanation or justification required.

As always, answers should be left in raw, unevaluated form, involving binomials, factorials, powers, etc. (such as $\binom{10}{3}2^3$). Do **not** use notations $C(n, k)$ or $P(n, k)$ in your answers.

All of the questions have a simple answer and require virtually no hand computations. Write legibly and circle or box your answer.

- (a) Write down an **exact** formula for D_n , the number of derangements of n objects.

Solution:
$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

- (b) What is the **approximate** probability that a random permutation of 10 objects has **exactly** 3 fixed points? (Your answer should be in terms of a famous constant.)

Solution: $\frac{1}{3!}e^{-1}$ This is a variation of 7.6:15 from HW 8. The **exact** value of the probability is $\frac{1}{10!} \binom{10}{3} D_7 = \frac{1}{3!7!} D_7$, since there are $\binom{10}{3}$ ways to choose the 3 fixed points and D_7 ways to derange the remaining 7 points. Since $D_7 \approx 7!e^{-1}$, the above is **approximately** equal to $\frac{1}{3!}e^{-1}$.

- (c) What is the total number of relations on the set $A = \{1, 2, 3, 4, 5\}$?

Solution: 2^{25} Since a relation is a subset of $A \times A$, the number of relations is equal to the number of subsets of $A \times A$. Since $A \times A$ has $5 \cdot 5 = 25$ elements, there are 2^{25} such subsets.

- (d) What is the number of relations on the set $A = \{1, 2, 3, 4, 5\}$ that are **symmetric and reflexive**?

Solution: $2^{\binom{5}{2}} = 2^{10}$ This was covered in class. The reasoning is as follows: Reflexivity means that all pairs (x, x) are in R , and symmetry means that (x, y) is in R if and only if (y, x) is in R . Thus, a symmetric and reflexive relation is completely determined by specifying which of the *unordered* pairs $\{x, y\}$ of *distinct* elements of A are in R . Since A has 5 elements, there are $\binom{5}{2}$ such pairs and hence $2^{\binom{5}{2}}$ ways to specify a subset of these pairs.

- (e) How many terms are in the inclusion-exclusion formula for a union of 10 sets? (No need to state the formula—just state the number of terms in it!)

Solution:
$$\sum_{k=1}^{10} \binom{10}{k} = 2^{10} - 1$$

There are $\binom{10}{1}$ terms involving “singles” A_i , $\binom{10}{2}$ terms involving “doubles” $A_i \cap A_j$, etc.; summing these numbers for $k = 1, \dots, 10$ gives the total number of terms.

- (f) What is the number of **onto** functions from the set $\{1, 2, 3, 4, 5\}$ to the set $\{1, 2, 3\}$? (This answer can involve a sum of a handful of terms.)

Solution:

$$3^5 - \binom{3}{1}(3-1)^5 + \binom{3}{2}(3-2)^5$$

2. For each of the given terms, state its *precise mathematical definition*, using correct set-theoretic notation, and including any necessary logical quantifiers and connectors (“for all”, “implies”, “if ... then”). The definitions should **not** be in terms of the graphical representations of relations.

- (a) A **relation** R on a set A is ... (Give the formal *set-theoretic* definition.)

Solution: a subset of $A \times A$

- (b) A relation R on a set A is **antisymmetric** if ...

Solution: If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

An equivalent definition is: If $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$

- (c) A **partition** of a set A is a collection of nonempty subsets A_i of A satisfying ... (State the properties that define a partition.)

Solution: The properties defining a partition of A are:

(i) $A_i \cap A_j = \emptyset$ for all $i \neq j$ (Pairwise disjoint)

(ii) $A_1 \cup A_2 \cup A_3 \cup \dots = A$ (Union is all of A)

- (d) Let A be a set, and $A_1, A_2, A_3 \dots$ be a partition of A into nonempty sets A_i . Define an equivalence relation R on A (by specifying the precise meaning of “ $(a, b) \in R$ ”) such that the equivalence classes for this relation are exactly the sets A_1, \dots, A_n . (Just give the precise definition for “ $(a, b) \in R$ ”. No explanation required.)

Solution: $(a, b) \in R \iff a \text{ and } b \text{ belong to the same set } A_i$ This is the converse part of the connection between partitions and equivalence relations covered in class (and also in the text on p. 561).

3. Let $A = \{n \in \mathbb{Z} : n \geq 100\}$ (i.e., A is the set of all positive integers having at least three digits in their decimal representation), and consider the relation R on A defined by $(a, b) \in R$ if the numbers a and b agree in their first three decimal digits.

- (a) Determine the equivalence class of 2010. (Your answer must be clear, unambiguous, and use proper mathematical notation; you can use dots (...), but only if the pattern of the sequence is completely clear and unambiguous.)

Solution: (This is a variation on 8.5:30 from HW 9) The equivalence class of 2010 is the *set of all positive integers whose decimal representation starts with 201*, i.e., the set

$$[2010] = \{201, 201*, 201**, 201***, \dots\},$$

where $*$ stands for an arbitrary decimal digit.

Note: A crucial point here is that the equivalence class is determined by the first three digits 201 only, and the 4th digit 0 in 2010 is irrelevant, as far as the equivalence class is concerned. An answer such as $\{2010, 20100, 20101, \dots\}$ would completely miss this key point.

- (b) Determine the number of equivalence classes of this relation.

Solution: The number of equivalence classes is equal to the number possible strings of 3 leading digits. There are 9 choices for the first digit (since it cannot be 0), and 10 each for the second and third digits. Hence there are $\boxed{9 \cdot 10 \cdot 10 = 900}$ equivalence classes.

4. Let a_n denote the number of ways to write n as a sum of 3's and 7's, **with order taken into account**. (So, for example, the representations $27 = 7 + 7 + 3 + 3 + 7$ and $27 = 3 + 7 + 7 + 3 + 7$ are considered different when computing a_{27}).

- (a) Set up a recurrence for a_n ; be sure to clearly explain how each term in this relation arises.

Solution: This is of the same type as the various stair/ladder climbing problems discussed in class and the homework. We classify the representations of n as a sum of 3's and 7's according to the *last* number appearing in the representation. That number is either 3 or 7.

- If the representation ends with a 3, the remaining terms in the representation give a representation of the desired form for $n-3$. There are a_{n-3} such representations.
- If the representation ends with a 7, the remaining terms in the representation give a representation of the desired form for $n-7$. There are a_{n-7} such representations.

The total number of representations is the sum of these two counts, i.e., we get the recurrence relation $\boxed{a_n = a_{n-3} + a_{n-7}}$.

- (b) How many initial terms need to be specified in order for the sequence to be uniquely determined by the recurrence and the initial conditions? Justify your answer. (You only need to find the *number* of initial terms needed, not the actual values.)

Solution: The number of initial terms needed is equal to the degree of the recurrence. Since the recurrence has degree 7, $\boxed{7}$ initial terms are needed.

5. The following questions are independent of each other.

- (a) Find the **general** solution to the recurrence $a_n = 8a_{n-1} - 16a_{n-2}$.

Solution: The associated characteristic equation is $r^2 - 8r + 16 = 0$, which has a double root $r = 4$. Thus, the fundamental solutions are 4^n and $n4^n$, and the general solution is of the form $\boxed{a_n = c_1 4^n + c_2 n 4^n}$.

- (b) Find the **general** solution to the recurrence $a_n = 8a_{n-1} + 9a_{n-2}$.

Solution: The associated characteristic equation is $r^2 - 8r - 9 = 0$, which has roots -1 and 9 . Thus, the fundamental solutions are $(-1)^n$ and 9^n , and the general solution is of the form $\boxed{a_n = c_1 (-1)^n + c_2 9^n}$.

- (c) Find a **particular** solution to the recurrence $a_n = 8a_{n-1} + 9a_{n-2} + 16n$.

Solution: The nonhomogeneous term is $16n$, so we seek a solution of the form (*) $a_n = An + B$ with constants A and B . Substituting (*) into the recurrence, we get

$$An + B = 8(A(n-1) + B) + 9(A(n-2) + B) + 16n = n(17A + 16) + (-26A + 17B).$$

Equating coefficients of n and of 1 (i.e., the constant terms) on both sides, we get the equations $A = 17A + 16$ and $B = -26A + 17B$, so $A = -1$ and $B = (26/16)A = 13/8$.

Thus, a particular solution is $\boxed{a_n = -n + \frac{13}{8}}$.

6. The following questions are independent of each other.

- (a) Let a_0, a_1, a_2, \dots be a sequence with generating function $G(x) = \sum_{n=0}^{\infty} a_n x^n$, and let b_0, b_1, b_2, \dots be the sequence whose generating function is $F(x) = G(3x^2)$. Find the latter sequence explicitly; i.e., obtain a general formula for the n -th term b_n , in terms of the a_k 's.

Solution: We have

$$F(x) = G(3x^2) = \sum_{k=0}^{\infty} a_k (3x^2)^k = \sum_{k=0}^{\infty} 3^k a_k x^{2k}.$$

Then b_n is the coefficient of x^n in the latter series, i.e.,

$$b_n = a_{n/2} 3^{n/2} \text{ if } n \text{ is even, and } b_n = 0 \text{ if } n \text{ is odd.}$$

Note: The b_n 's are *coefficients*, so they should not involve powers of x . An answer like $b_k = 3^k a_k x^k$ obtained by trying to "split off" a power x^k in the above series is nonsensical.

- (b) Now consider the particular sequence defined by the initial conditions $a_0 = 4$ and $a_1 = 3$ and the recurrence relation $a_n = 5a_{n-1} + 6a_{n-2}$ for $n = 2, 3, \dots$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for this sequence. Derive an explicit formula for $G(x)$ (as a rational function of x , e.g., $15x^2/(1+x+x^2)$) **directly from the given recurrence and initial conditions** (i.e., without using formulas for a_n derived by other methods). (No need to solve the recurrence. Just find the generating function.)

Solution: This is of the same type as the generating function solution to the Fibonacci recurrence worked out in class, and Problem 7.4.35 from HW 7. To obtain $G(x)$, multiply the recurrence $a_n = 5a_{n-1} + 6a_{n-2}$ by x^n , sum over $n = 2, 3, 4, \dots$, express the resulting identity in terms of $G(x)$, and solve for $G(x)$.

$$\begin{aligned} \sum_{n=2}^{\infty} a_n x^n &= 5 \sum_{n=2}^{\infty} a_{n-1} x^n + 6 \sum_{n=2}^{\infty} a_{n-2} x^n, \\ \sum_{n=2}^{\infty} a_n x^n &= 5x \sum_{m=1}^{\infty} a_m x^m + 6x^2 \sum_{k=0}^{\infty} a_k x^k \quad (\text{set } m = n-1, k = n-2), \\ G(x) - a_0 - a_1 x &= 5x(G(x) - a_0) + 6x^2 G(x), \\ G(x) &= \frac{a_0 + x(a_1 - 5a_0)}{1 - 5x - 6x^2} = \frac{4 - 17x}{1 - 5x - 6x^2} \end{aligned}$$

Note: An absolutely crucial point in this argument, pointed out in class and in the hw solutions, are the ranges of summation in the various sums. In particular, since the recurrence is only valid for $n = 2, 3, \dots$, one cannot start the summation at $n = 0$.