Problem 1

Let

$$f(n) = n(n+1) + (n+1)(n+2) + \cdots + (2n-1)(2n)$$

Using the definition of the Big-oh notation, prove that the function $f(n)$ is $O(n^3)$. Provide (with proof) explicit values of the witnesses C and k.

Solution: We will show that

$$(*) \qquad |f(n)| \le 4n^3 \quad (n \ge 1),$$

i.e., that the required Big-oh relation holds with witnesses $\boxed{C = 4}$ and $\boxed{k = 1}$. To prove $(*)$, note that each term in the above sum is less than $(2n)(2n) = 4n^2$, and that there are n such terms in this sum. Thus, the sum $f(n)$ is less than $n \cdot 4n^2 = 4n^3$, as claimed in $(*)$.

Problem 2

Determine the number of ways to arrange the 11 letters of the word MISSISSIPPI

(i) without restrictions.

Solution: We have 4 S's, 4 I's, 2 P's, and 1 M to place into 11 slots. There are $\binom{11}{4}$ ways to pick slots for the 4 letters S, $\binom{7}{4}$ ways to pick slots for the 4 letters I, $\binom{3}{2}$ ways to pick slots for the 2 letters P, and $\binom{1}{1} = 1$ way to pick a slot for the remaining letter M, giving a total of

$$\boxed{\binom{11}{4}\binom{7}{4}\binom{3}{2}}$$

Alternative approach: Using multinomial coefficients, the number of rearrangements is $\frac{11!}{4!4!3!1!}$. (It can be checked that this is the same as the above product of binomial coefficients.)

(ii) if identical letters must be adjacent, as in IIIIPPSSSSM.

Solution: There are four different letters in the word, and since identical letters must appear in blocks, an arrangement of the required form is equivalent to a permutation of these four letters. There are $\boxed{4! = 24}$ such permutations.

Problem 3

Evaluate the sum

$$\binom{100}{0} - 2\binom{100}{1} + 2^2\binom{100}{2} - \cdots - 2^{99}\binom{100}{99} + 2^{100}\binom{100}{100}.$$

Solution: By the binomial theorem, the given sum is

$$\sum_{k=0}^{100} (-2)^k \binom{100}{k} = \boxed{(1-2)^{100} = 1}$$

Problem 4

A test has 5 questions, each with possible scores of 0, 1, 2, or 3 points.

(i) What is the minimal number of students in a class needed in order to **guarantee** that there are two students with identical score sheets? (A score sheet shows the number of points the student obtained on each question, just like the table on the cover page of this exam. Thus, two students have identical score sheets if and only if they received the same score on each of the questions.)

Solution: A score sheet can be represented by a tuple $(s_1, s_2, s_3, s_4, s_5)$, with $s_i \in \{0, 1, 2, 3\}$. There are $4^5 = 2^{10} = 1024$ such tuples, so there are 1024 possible score sheets. By the pigeonhole principle, the minimal number of students needed to guarantee two identical score sheets is one more than this number, i.e., $\boxed{1025}$.

(ii) What is the probability that, in a class of 40, there are at least two students with the same score sheets?

Solution: This is a birthday type problem with the 1024 possible score sheets corresponding to the 365 possible birthdays. The probability that all score sheets are distinct is $(1024 \cdot 1023 \ldots 995)/1024^{40}$, and the probability that at least two score sheets are identical is 1 minus the above, i.e.,

$$\boxed{1 - \frac{1024 \cdot 1023 \ldots 985}{1024^{40}} \; (= 0.5377\ldots)}.$$

Remark: Thus, in a class of 40 there is already a greater than 50-50 chance that two score sheets are identical. That only 40 students should be needed for this is surprising, but consistent with the birthday paradox.

Problem 5

Let $F_n = \sum_{i=1}^{n} f_i$, where $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, \ldots$ are the Fibonacci numbers. Guess a formula for F_n, and then use induction to prove that this formula holds for all positive integers n. Your write-up of the induction proof must be mathematically rigorous and complete, be presented in logical order with all necessary steps and explanations where needed (e.g., say things like “by the definition of . . . “, “by the induction hypothesis”, or “by algebra” at the appropriate places). A list of disconnected formulas does not constitute a proof.

Solution: The first few values of F_n are $F_1 = 1$, $F_2 = 2$, $F_3 = 4$, $F_4 = 7$, $F_5 = 12$, $F_6 = 20$. Note that these numbers are 1 less than the Fibbonnaci numbers with an index shifted by 2. This suggests the general formula

$$(*) \qquad F_n = f_{n+2} - 1.$$

We prove this by induction.

Base step: For $n = 1$, we have $F_1 = 1 = 2 - 1 = f_3 - 1$, so $(*)$ holds in this case.

Inductive step: Assume $(*)$ holds for $n = k$, where k is a positive integer. Then

$$\begin{aligned} F_{k+1} &= \sum_{i=1}^{k+1} f_i = \sum_{i=1}^{k} f_i + f_{k+1} = F_k + f_{k+1} \quad \text{(by def. of } F_{k+1} \text{ and } F_k) \\ &= f_{k+2} - 1 + f_{k+1} \quad \text{(by ind. hyp.)} \\ &= f_{k+3} - 1 \quad \text{(by the recurrence for } f_n). \end{aligned}$$

This proves that $(*)$ holds for $n = k + 1$, so the induction is complete.

Problem 6

The following are poker problems. Recall that there are 13 possible kinds of cards, and a poker deck consists of 52 cards, with four cards of each kind, one for each of the four suits.

(i) What is the probability that in a 5-card poker hand all 5 cards are from the same suit?

Solution: The total number of 5 card poker hands is $\binom{52}{5}$. There are 4 ways to pick a suit, and $\binom{13}{5}$ ways to choose 5 differents kinds from this suit. Thus, the number of poker hands with all cards from the same suit is $4\binom{13}{5}$, and the probability for such a hand is

$$\boxed{\frac{4\binom{13}{5}}{\binom{52}{5}}}.$$

(ii) What is the probability that in a 5-card poker hand all 5 cards are of different kinds?

Solution: There are $\binom{13}{5}$ ways to choose 5 differents kinds, and for each of these 5 kinds there are 4 ways to choose a suit. Thus, the total number of poker hands with 5 different kinds is $\binom{13}{5}4^5$, and its probability is

$$\boxed{\frac{\binom{13}{5}4^5}{\binom{52}{5}}}.$$

Problem 7

A bank account earns 5% annual interest, which is deposited at the end of each year. Suppose you open such an account with an initial deposit of 100 dollars, and that at the end of each year you deposit an additional 100 dollars to the account. Let a_n denote the value of the account at the end of n years. Find a formula for a_n. (The answer should be a simple function of n; it should not involve summations.)

Solution: The given information translates into the recurrence $a_{n+1} = 1.05a_n + 100$, with initial condition $a_0 = 1$. We solve this recurrence by iteration:

$$\begin{aligned} a_n &= 1.05a_{n-1} + 100 = 1.05(1.05a_{n-2} + 100) + 100 = 1.05^2 a_{n-2} + (1.05 + 1)100 \\ &= \cdots = 1.05^n a_0 + (1.05^{n-1} + 1.05^{n-2} + \cdots + 1)100 \\ &= \boxed{\left(1.05^n + \frac{1.05^n - 1}{1.05 - 1}\right) 100 (= (1.05^n 2100 - 2000)} \end{aligned}$$

Problem 8

(i) Evaluate the generalized binomial coefficient $\binom{-1/3}{3}$. (Here a concrete answer is expected, in the form of a single rational number.)

Solution: By the definition of generalized binomial coefficients,

$$\binom{-1/3}{3} = \frac{(-1/3)(-4/3)(-7/3)}{3!} = \frac{-1\cdot 4\cdot 7}{3^3\cdot 6} = \boxed{\frac{-14}{81}}.$$

(ii) Evaluate the infinite series $\sum_{n=0}^{\infty}\binom{-1/3}{n}x^n$ in closed form, i.e., express the sum of this series in terms of a simple elementary function.

Solution: By the extended binomial theorem,

$$\sum_{n=0}^{\infty}\binom{-1/3}{n}x^n = \boxed{(1+x)^{-1/3}}.$$

Problem 9

Find the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 20,$$

under the restriction that the x_i's be integers satisfying the constraints

$$x_1 \ge 1, \quad x_2 \ge 2, \quad x_3 \ge 3, \quad x_4 \ge 4.$$

Solution: Setting $x_i = x_i' + i$ for $i = 1, 2, 3, 4$, the equation becomes

$$x_1' + x_2' + x_3' + x_4' = 20 - 1 - 2 - 3 - 4 = 10,$$

and the restrictions on the x_i become $x_i' \ge 0$, i.e., the x_i' can be arbitrary nonnegative integers. The latter equation has $\binom{10+4-1}{10} = \boxed{\binom{13}{10}}$ solutions.

Problem 10

How many positive integers less than or equal to 1000 are there which are divisible by **none of** the numbers 5, 7, and 11? Your answer can be left in "raw" form and involve expressions like $\lfloor \frac{213}{2\cdot 13} \rfloor$.

Solution: By inclusion/exclusion, the number sought is

$$1000 - \left\lfloor\frac{1000}{5}\right\rfloor - \left\lfloor\frac{1000}{7}\right\rfloor - \left\lfloor\frac{1000}{11}\right\rfloor + \left\lfloor\frac{1000}{5\cdot 7}\right\rfloor + \left\lfloor\frac{1000}{5\cdot 11}\right\rfloor + \left\lfloor\frac{1000}{7\cdot 11}\right\rfloor - \left\lfloor\frac{1000}{5\cdot 7\cdot 11}\right\rfloor$$

Problem 11

Consider the relation R on the set of **nonzero real numbers** defined by $(x, y) \in R$ if and only if $xy < 0$. Determine which (if any) of the three properties "reflexive", "symmetric", and "transitive" this relation has. In each case, if the relation has the property, explain why; otherwise, give a concrete counter-example.

Solution: The relation is **not reflexive**, since (for example), $(2,2) \notin R$ since $2 \cdot 2 \not< 0$.

It is **symmetric**, since $xy < 0$ holds if and only if $yx < 0$.

It is **not transitive**, since $(1,-1) \in R$, $(-1,1) \in R$, but $(1,1) \notin R$.

Problem 12

How many (binary) relations are there on a set A with 5 elements which are *reflexive and symmetric*?

Solution: A relation A is any subset of $A \times A$. A reflexive relation must contain all pairs (a,a) with $a \in A$, and a symmetric relation contains (a,b) if and only if it contains (b,a). Thus, a reflexive and symmetric relation on A is uniquely determined by specifying which *unordered* pairs $\{a,b\}$ of distinct elements in A it contains. Since A has 5 elements, there are $\binom{5}{2} = 10$ such pairs, and hence $\boxed{2^{10} = 1024}$ ways to pick a subset of these.

Alternative solution: If we represent a relation on A by a binary 5×5 matrix, the reflexive and symmetric relations correspond to matrices that have 1's on the diagonal and are symmetric. These matrices are uniquely determined by their entries on the upper right triangle, excluding the diagonal. There are $4+3+2+1 = 10$ such entries, and hence 2^{10} ways of specifying their binary values.