Math 213, Spring 2006
Generating Functions

Some useful power series expansions

1. $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ (geom. series)

2. $\displaystyle\sum_{k=n}^{\infty} x^k = \frac{x^n}{1-x}$ (tail of geom. series)

3. $\displaystyle\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ (finite geom. series)

4. $\displaystyle\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$ (derivative of geom. series, multiplied by x)

5. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$ (binomial theorem)

6. $\displaystyle\sum_{k=0}^{\infty} \binom{u}{k} x^k = (1+x)^u$ (u any real number; $\binom{u}{k} = \dfrac{u(u-1)\dots(u-k+1)}{k!}$)
   (extended binomial theorem; binomial series)

7. $\displaystyle\sum_{k=0}^{\infty} \binom{n+k}{n} x^k = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k = \frac{1}{(1-x)^{n+1}}$

8. $\displaystyle\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k = \frac{1}{(1-x)^n}$

9. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ (exponential series)

10. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = \ln(1+x)$ (logarithmic series)

11. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\frac{1}{1-x}$ (logarithmic series, variant)

Some manipulations with generating functions

In the formulas below, $F(x) = \sum_{k=0}^{\infty} a_k x^k$ and $G(x) = \sum_{k=0}^{\infty} b_k x^k$.

1. $F(x) + G(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ (addition)

2. $F(x)G(x) = \sum_{k=0}^{\infty} c_k x^k$, where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ (multiplication)

3. $z^m F(x) = \sum_{k=m}^{\infty} a_{k-m} x^k$ ($m \geq 0$ integer) (shifting index)

4. $F'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ (differentiation)

5. $\int_0^x F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ (integration)