More tips on writing up induction proofs

- **Structure of an induction proof:** Each proof must contain (1) a precise statement of the proposition to be proved, (2) a proof of the base case, (3) a proof of the induction step, and (4) a conclusion. The proof is incomplete if any of these parts is missing.
- **Start out any induction proof with a precise statement of the proposition/ formula/statement to be proved.** A good idea is to display this statement and give it a label (e.g., $P(n)$, or $(*)$, so you can refer to it later.
- **Start out the induction step with a precise statement of the induction hypothesis, i.e., what is being assumed in the proof of the induction step.** Without an explicitly stated assumption, the argument is incomplete. The appropriate induction hypothesis depends on the nature of the problem and the type of induction used. Here are some common ways to start out an induction step:
  - "Let $k \in \mathbb{Z}_+$ and assume $(*)$ is true for $n = k$."
  - "Let $k \geq 2$ and assume $(*)$ holds for $n = k - 1$ and $n = k$."
  - "Let $k \geq 1$ and assume $P(n)$ holds for $n = 1, 2, \ldots, k$."

  This assumes that the statements referred to, $(*)$ or $P(k)$, have been precisely defined.
- **Distinguish between the variable in the statement to be proved (typically denoted by "n") and the "running" variable that occurs in the induction step (usually denoted by "k").** This allows you to say something like the following: "Let $k \in \mathbb{Z}_+$ be given, and suppose (1) is true for $n = k$. .... [Proof of induction step goes here] ... Therefore (1) is true for $n = k + 1$."
- **In the proof of the induction step, justify any key step (e.g., by annotations like ("by (3)" or "by the assumption .."). Most importantly, clearly indicate at which point in the proof the induction hypothesis is used (e.g., "by induction hypothesis (*) with $n = k - 1$").** Getting the induction hypothesis into play is the most critical part of any induction proof; if you do not use the induction hypothesis, something is wrong with your argument.

Strong induction (Rosen, Section 4.2)

Sometimes, in trying to get the $k + 1$ case to work out, you may find that, in addition to assuming the case k (which is the hypothesis in the standard induction argument), you'll also need to assume earlier cases, such as the case $k - 1$, and possibly all cases from $1, 2, \ldots, k$. Since all of these cases have already been encountered on the way to the case k, this additional assumption can safely be made, i.e., instead of assuming just "$P(k)$" in the induction step, we may assume "$P(1)$ and $P(2)$ and ... and $P(k)$". This variant of an induction proof is called "strong induction."

A standard application of strong induction (with the induction hypothesis being "$P(k-1)$ and $P(k)$" instead of just "$P(k)$") is to proving identities and relations for Fibonacci numbers and other recurrences. The Fibonacci sequence is defined by

$$f_0 = 0, f_1 = 1, f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad (n \geq 3).$$

Its first few terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. The Fibonacci sequence is the most famous sequence in mathematics; it's the only sequence that has its own journal ("Fibonacci Quarterly"), its own professional society ("Fibonacci Association"), and its own series of professional conferences (the 14th International Conference on Fibonacci Numbers and Their Applications took place in Morelia, Mexico, July 5–9, 2010). There are hundreds of formulas involving Fibonacci numbers, most of which quite routine to prove by induction or strong induction once you are given (or have guessed) the formula. The following illustrate this type of application.

Induction practice problems, II

(Solutions will be posted on the course webpage by early next week.)

3. **Identities for Fibonacci numbers and other recurrences.** See above for the definition of f_n. Some of these problems require "strong" induction.

   (a) Prove that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all $n \in \mathbb{Z}_+$.

   (b) Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$ for all $n \in \mathbb{Z}_+$.

   (c) Prove that $f_n \geq (3/2)^{n-2}$ for all $n \in \mathbb{Z}_+$.

   (d) Prove that $\sum_{i=1}^{n} f_{2i-1} = f_{2n}$ for all $n \in \mathbb{Z}_+$.

   (e) Let a_n be the sequence defined by $a_1 = 1, a_2 = 8, \quad a_n = a_{n-1} + 2a_{n-2} \quad (n \geq 3)$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{Z}_+$.

4. **Induction proofs: other types.** Here are some problems involving induction proofs that do not fall into one of the above types. The first is a classical illustration that you should be familiar with. The second example illustrates a common application of induction, namely to extend theorems or formulas that have been proved for the case of 2 variables to the case of n variables.

   (a) **Number of subsets (cf. Rosen, 4.1, Example 9):** Show that a set of n elements has 2^n subsets.

   (b) **De Morgan's Law for n sets (cf. Rosen, 4.1, Example 10):** Show that if $A_1, \dots, A_n$ are sets, then $\overline{(A_1 \cup \dots \cup A_n)} = \overline{A_1} \cap \dots \cap \overline{A_n}$. (This is a generalization of De Morgan's Law to n sets. Use De Morgan's Law for two sets ($\overline{A \cup B} = \overline{A} \cap \overline{B}$) and induction to prove this result.)

5. **Examples and practice problems from the Rosen text.** Induction proofs is very much a "learnable" skill, but to get proficient and comfortable with induction proofs requires lots of practice. There is no single method that can be applied to all problems, but the more problems you study and work out, the broader your experience becomes and the better you equipped you become in handling new situations. The Rosen text is a great source of examples and exercises on induction. The examples are all worked out in detail, and the odd-numbered exercises have solutions in the back of the book. The following are examples that you should study and exercises you should do, in addition to those above.

   - **Sum/product formulas:** Section 4.1, Examples 1–4, Exercises 3, 5, 7, 13, 15.
   - **Inequalities:** Section 4.1, Examples 5–6, Exercises 19, 21, 23, 25.
   - **Fibonacci identities:** Section 4.3, Example 6, Exercise 13, 15. Note that these problems are straight induction problems that do not require any of the material and concepts from Section 4.3 (which we will not cover in class).
   - **Other types of induction proofs:** Section 4.1, Examples 9, 10, Exercise 45.