Exam 2 Solutions and Comments

- **Grading:** Each problem was worth 20 points. The maximal total score is 140 points, and the average was 113 points.
- **Curving and letter grades.** The maximal number of points from the homework assignments given out so far and the two midterm exams is 424; you can check your point totals through the link at the course webpage. By the end of the semester, the maximal number of points will have reached 700–1000 points; I will then set a global curve at that time depending on the overall score distribution of the class. As a **rough** guide, here are approximate letter grade ranges for your current score totals: ¿350: A; 290–349: B; 230–289: C. (Note that I do give plusses and minuses, so each range comprises three subranges, A+, A, A-, etc.)
- **Solutions:** Detailed solutions, along with some grading comments, are attached. If you didn't get full credit on a problem, please check these solutions, and any comments there (e.g., about common errors), before asking what you did wrong.

#	1	2	3	4	5	6	7	Total
Points	20	20	20	20	20	20	20	140

Problem 1

Determine all local minima, local maxima, and saddle points of the function $f(x, y) = x^2 + y^2 + x^2y + 4$.

Solution: We first determine all critical points of f. The gradient of f is $\nabla f(x, y) = \langle 2x + 2xy, 2y + x^2 \rangle$. Setting this equal to the zero vector gives the equations for the critical points:

$$2x + 2xy = 0, \quad 2y + x^2 = 0.$$

To solve this system, we first use the second equation to get $y = -x^2/2$, and substitute this back into the first equation to get $2x + 2x(-x^2/2) = -x^3 + 2x = 0$, or $x^3 = 2x$, or $x(x^2 - 2) = 0$. The latter equation has three solutions: $x = 0$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Substituting these values into the first equation gives the y-values corresponding to these solutions: $y = 0$, $y = -1$, and $y = -1$. Hence the critical points are $(0, 0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$.

Next, we apply the second derivative test to each of these points in order to determine whether it is a saddle point, a local min or a local max. We compute:

$$f_{xx} = 2 + 2y, \quad f_{yy} = 2, \quad f_{xy} = f_{yx} = 2x,$$
$$\Delta = f_{xx}f_{yy} - f_{xy}^2 = (2 + 2y)(2) - (2x)^2 = 4 + 4y - 4x^2.$$

The condition for a saddle point is that $\Delta < 0$, for a local minimum/maximum it is $\Delta > 0$, with a minimum occurring if $f_{xx} > 0$ and a maximum if $f_{xx} < 0$. Substituting the three critical points into this formula, we get:

- $(0, 0)$: $\Delta = 4$ and $f_{xx} = 2 > 0$, so $\boxed{(0, 0) \text{ is a local minimum}}$
- $(\sqrt{2}, -1)$: $\Delta = -8$, so $\boxed{(\sqrt{2}, -1) \text{ is a saddle point}}$
- $(-\sqrt{2}, -1)$: $\Delta = -8$, so $\boxed{(-\sqrt{2}, -1) \text{ is a saddle point}}$

Remarks: Note that in solving the critical points equations one has to be careful not to divide by x since then one would miss out on the case $x = 0$, and the point $(0, 0)$. Also note that $x^2 = 2$ has two solutions, $x = \sqrt{2}$ and $x = -\sqrt{2}$, yielding two points, $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$.

Problem 2

Evaluate the integral $\int_{x=0}^{\infty}\int_{y=0}^{\sqrt{3}x} e^{-x^2}e^{-y^2}\,dydx$. (Hint: use polar coordinates.)

Solution: (This is a variation of the double integral that arises when evaluating the Gaussian integral. The only difference is in the θ-limits: θ ranges from 0 to $\pi/3$.) The region of integration is given by the inequalities $0 \le x < \infty$, $0 \le y \le \sqrt{3}x$, which is the part of first quadrant below the line $y = \sqrt{3}x$ (a wedge-shaped region). The line $y = \sqrt{3}x$ has slope $\sqrt{3}$, so it forms an angle of $\pi/3$ with the x-axis. Thus, in polar coordinates, the region described by the inequalities $0 \le r < \infty, 0 \le \theta \le \pi/3$. and the integral in polar coordinates is

$$\boxed{\int_{\theta=0}^{\pi/3}\int_{r=0}^{\infty} e^{-r^2}\,rdrd\theta}.$$

To evaluate the inner integral, use the substitution $u = r^2, du = 2rdr$:

$$\int_{r=0}^{\infty} e^{-r^2}rdr = \int_{u=0}^{\infty} e^{-u}(1/2)du = \left[(-1/2)e^{-u}\right]_{u=0}^{u=\infty} = -(-1/2)e^{-0} = \frac{1}{2}.$$

Substituting this into the above double integral gives the answer:

$$\int_{\theta=0}^{\pi/3}\frac{1}{2}d\theta = \frac{1}{2}\cdot\frac{\pi}{3} = \boxed{\frac{\pi}{6}}.$$

Problem 3

Find the mass of a solid sphere of radius a centered at the origin with mass density given by the function

$$\delta(x, y, z) = e^{-(x^2+y^2+z^2)^{3/2}}.$$

Solution: In spherical coordinates, the density is simply $\delta = e^{-\rho^3}$, and the sphere is described by the inequalities $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, $0 \leq \rho \leq a$. Thus

$$\begin{aligned}
m &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{a} e^{-\rho^3} \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin\phi \int_{u=0}^{a^3} e^{-u}(1/3) du d\phi d\theta \quad (\text{set } u = \rho^3,\ du = 3\rho^2 \, d\rho) \\
&= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin\phi (1/3)(1 - e^{-a^3}) d\phi d\theta \\
&= \int_{0}^{2\pi} \int_{0}^{\pi} [-\cos\phi]_{\phi=0}^{\pi} (1/3)(1 - e^{-a^3}) d\theta \\
&= (2\pi)(2)(1/3)(1 - e^{-a^3}) = \boxed{\frac{4\pi}{3}(1 - e^{-a^3})}
\end{aligned}$$

Remarks. If this looked vaguely familiar, it's no accident: the above triple integral is the same as that of Problem 7 in 13.7, a hw problem.

Note that this problem **cannot be done with double integrals**. Only volumes can be computed (in certain cases) with double integrals, but the given problem asks for the mass of a three-dimensional object with a mass density function $\delta(x, y, z)$. The triple-integral approach is the only option here.

The problem **cannot be done with cylindrical or rectangular coordinates**. Using spherical coordinates is the only way to evaluate the integral. Because of the shape of the object (a sphere), spherical coordinates are in fact the obvious, and most natural, choice. If one tries to use cylindrical coordinates one ends up with integrals that cannot be evaluated.

Problem 4

Let T denote the tetrahedron (pyramid) with corners $(0,0,0)$, $(2,0,0)$, $(0,3,0)$, $(0,0,6)$. Set up (but do not evaluate) the volume of T as a **triple** integral in **rectangular** coordinates, i.e., in the form $\int_*^* \int_*^* \int_*^* * \, dzdydx$, with appropriate expressions in place of the 7 asterisks.

Solution: Sketching the tetrahedron, we see that the coordinate planes in the first octant fom three of its sides, while the fourth side, the "roof", is given by the plane through the points $A = (2,0,0)$, $B = (0,3,0)$, $C = (0,0,6)$. To obtain the equation of this plane, we first find a normal vector:

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = 18\vec{i} - (-12)\vec{j} + 6\vec{k} = \langle 18, 12, 6 \rangle.$$

Using $\langle 18, 12, 6 \rangle$ as normal vector and $(2,0,0)$ as point on the plane, we get $(x-2)18 + (y-0)12 + (z-0)6 = 0$ or $\boxed{z = 6 - 3x - 2y}$, as the equation of the roof.

The "floor" of the tetrahedron is the triangle determined by the points $(0,0)$, $(2,0)$ and $(3,0)$ in the xy-plane. A sketch shows that this is the region R between $x = 0$ and $x = 2$, bounded above by the line $y = 3 - (3/2)x$, below by the $y = 0$; i.e., R is described by the inequalities

$$0 \le x \le 2, \quad 0 \le y \le 3 - (3/2)x.$$

Tacking on the bounds for z derived above, $z_{BOT} = 0$ (for the floor of the tetrahedron) and $z_{TOP} = 6 - 3x - 2y$ (for the roof), we get inequalities for x, y, and z that describe T:

$$0 \le x \le 2, \quad 0 \le y \le 3 - (3/2)x, \quad 0 \le z \le 6 - 3x - 2y.$$

Thus the triple integral representing the volume of T is

$$V(T) = \iiint_T 1 dV = \int_{x=0}^{2} \int_{y=0}^{3-(3/2)x} \int_{z=0}^{6-3x-2y} 1 \, dz \, dy \, dx.$$

Remarks: The tetrahedron here is the same as that in Problem 27 of 13.3. However, in contrast to the latter problem, the equation of the plane that constitutes the "roof" was not given, but had to be first derived. The correct derivation of this equation is a key part of the solution to this problem.

Note that in the **triple integral formula** for a volume (and this is the formula sought in the problem) the integrand is 1; it is only in the **double integral formula** that a function appears in the integrand (namely the function $z_{TOP} - z_{BOT}$, where z_{TOP} and z_{BOT} are the functions of x and y representing the "roof" and the "floor" of the solid.

Problem 5

Find I_z, the **moment of inertia about the z-axis**, for the solid that is bounded from below by the paraboloid $z = \frac{1}{2}(x^2 + y^2)$ and from above by the plane $z = 2$, assuming the mass density is equal to 1.

Solution: Note that the plane $z = 2$ intersects the paraboloid when $2 = (1/2)(x^2 + y^2)$, which is a circle of radius 1 about the origin. Thus, the region R in the xy-plane over which the given solid lies is the unit disk centered at the origin, which in polar coordinates is simply $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$. The full solid is described in cylindrical coordinates by the inequalities

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad (1/2)r^2 \leq z \leq 2.$$

The moment of inertia is therefore

$$\begin{aligned} m &= \iiint_E (x^2 + y^2)\delta dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r^2/2}^{2} r^2 r dz dr d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left(2 - \frac{r^2}{2}\right) r^3 dr d\theta \\ &= \int_{\theta=0}^{2\pi} \left(2 \cdot \frac{2^4}{4} - \frac{1}{2} \cdot \frac{2^6}{6}\right) d\theta = \int_0^{2\pi} \left(8 - \frac{16}{3}\right) d\theta = \boxed{\frac{16\pi}{3}} \end{aligned}$$

Problem 6

(i) Compute the Jacobian determinant of the transformation $x = u^2$, $y = v^2$, $z = w^2$.

Solution: The Jacobian of this transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = \boxed{8uvw}.$$

(ii) Using the transformation given in (i), set up (but do not evaluate) a triple integral in the variables u, v, w, that gives the volume of the region (in the first octant) that is bounded by the surface $\sqrt{x}+\sqrt{y}+\sqrt{z} = 1$ and the coordinate planes. Your answer should be of the form $V = \int_*^* \int_*^* \int_*^* * \, dwdvdu$, with appropriate expressions in place of the 7 asterisks.

Solution: Setting $u = \sqrt{x}$, $v = \sqrt{y}$, $w = \sqrt{z}$, the given surface becomes $u+w+w = 1$, and the region bounded by this surface and the coordinate planes (i.e., the planes $x = 0$, $y = 0$, $z = 0$, or equivalently $u = 0$, $v = 0$, $w = 0$) is given by the inequalitites $0 \le u \le 1$, $\quad 0 \le v \le 1-u$, $\quad 0 \le w \le 1-u-w$. Keeping in mind the extra factor from the Jacobian determinant computed in the first part, $8uvw$, the integral giving the volume is

$$V = \boxed{\int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dwdvdu}$$

Problem 7

The instructor of a certain math class gives students the option to have their overall score (which determines course grade) computed via either of the two formulas,

(A) $$A = A(x, y, z) = (1/2)x + (1/3)y + (1/6)z$$

and

(G) $$G = G(x, y, z) = x^{1/2}y^{1/3}z^{1/6},$$

where x, y, and z represent the (numerical) scores on midterm exams, the final exam, and the homework. (For the purposes of this problem, assume that the scores x, y, z are nonnegative real numbers, but otherwise unrestricted.) Which of the two formulas ((A) or (G)) should you choose if you want to maximize your overall score?

To answer this question, **use the Lagrange multipler method** to show that one of the two formulas ((A) or (G) — you need to determine which) **always** beats the other, in the sense that is produces a value that is greater than or equal to that obtained by the other formula, regardless of what the scores x, y, and z are. Here are some tips:

(1) Fix one of the expressions $A(x, y, z)$ or $G(x, y, z)$ (you decide which) at a constant value, say c.

(2) When applying the Lagrange multiplier method, clearly state the function you want to optimize (maximize or minimize—state which) and the constraint equation.

(3) A key part to solving this problem is to show that the solution you found represents a maximum and not a minimum (or vice versa, if you are looking for a minimum); after all, this determines which of the two formulas you should choose. This requires an ironclad, rocksolid, bulletproof argument; an educated guess, or a plausibility argument, won't do!

(4) If you need additional space for the solution, use the back of this page.

Solution: This is an application of the Lagrange multiplier method to proving a mathematical inequality. We will show that

(1) $$A(x, y, z) \geq G(x, y, z)$$

for all $x, y, z \geq 0$. Thus, formula (A) always results in a larger (or at worst equal) value than formula (G), so (A) is the way to go!

To prove (1), we proceed in two stages: The first step is to solve the following optimization problem:

(2)

Optimize the function $A(x, y, z)$ subject to the constraint $G(x, y, z) = c$.

Here c is an arbitrary *positive* constant. (In the case $c = 0$, we have $G(x, y, z) = 0$. On the other hand, since the numbers x, y, and y are nonnegative, $A(x, y, z)$ is always ≥ 0. Thus (1) holds in this case, and so it suffices to consider the case when $c > 0$.)

The second step consists of showing that the extremal value found is a minimum, and not a maximum.

Step I: Solution of the optimization problem (2): The Lagrange equations for this problem are $\nabla A = \lambda \nabla G$, $G(x, y, z) - c = 0$. Computing the gradients of the functions A and G, we get

$$\begin{aligned} \frac{1}{2} &= \lambda \frac{1}{2} x^{-1/2} y^{1/3} z^{1/6}, \\ \frac{1}{3} &= \lambda \frac{1}{3} x^{1/2} y^{-2/3} z^{1/6}, \\ \frac{1}{6} &= \lambda \frac{1}{6} x^{1/2} y^{1/3} z^{-5/6}, \\ 0 &= x^{1/2} y^{1/3} z^{1/6} - c. \end{aligned}$$

A painless way to solve this system is to multiply the first three equations by $2x$, $3y$, and $6z$, respectively. Then the right-hand sides in these equations become $\lambda x^{1/2} y^{1/3} z^{1/4} = \lambda G(x, y, z) = \lambda c$, and the left-hand sides x, y and z, so we get $x = y = z = \lambda c$. Plugging these values into the last equation gives

$$(\lambda c)^{1/2} (\lambda c)^{1/3} (\lambda c)^{1/6} = c,$$

which implies first $\lambda = 1$, so the point at which the extremum is attained is

$$\boxed{x = c,\ y = c,\ z = c}$$

Finally, plugging these values into $A(x, y, z)$ we get $A(x, y, z) = (1/2)c + (1/3)c + (1/6)c = c$ as the extremal value. However this is not the end of the story, as we do not yet know whether c is a maximum or a minimum.

Step II: Proof that the extremum found in Step I is a minimum, and not a maximum. The above calculations and the result obtained (i.e., $(x, y, z) = (c, c, c)$ as the extremal point and $A(x, y, z) = c$ as the extremal value) are exactly the same, whether we set out to find a maximum or a minimum. Thus, these calculations by themselves do not show that we have a minimum. (Of course, simply using the word “minimize” instead of “optimize” in the statement of the Lagrange problem does not prove this either.)

Therefore an additional argument is needed. A clearcut way to rule out the possibility that the point found is a maximum is to show that such a maximum does not exist, i.e., that $A(x, y, z)$ *can become arbitrarily large subject to the constraint* $G(x, y, z) = c$. This is what we will do. Take

$$x = Kc, \quad y = \frac{c}{K}, \quad z = \frac{c}{K},$$

where c is the constant in the constraint equation and K is a constant that is at our disposal and which we can make arbitrarily large. With this choice,

$$G(x, y, z) = (Kc)^{1/2} \left(\frac{c}{K}\right)^{1/3} \left(\frac{c}{K}\right)^{1/6} = c,$$

so the constraint is satisfied. On the other hand,

$$A(x, y, z) = \frac{cK}{2} + \frac{c}{3K} + \frac{c}{6K} > \frac{cK}{2},$$

and this can be made arbitrarily large with an appropriate choice of K. Therefore, under the constraint $G(x, y, z) = c$, $A(x, y, z)$ does not have a (finite) maximum. Hence, the extremal value c found in Step I must represent a minimum of $A(x, y, z)$ subject to the constraint $G(x, y, z) = c$.

Step III: Conclusion. From Steps I and II we conclude that c is the *minimal* value of $A(x,y,z)$ among choices of x,y,z satisfying the constraint $G(x,y,z)=c$. Hence, for all such x,y,z, the value of $A(x,y,z)$ must be greater or equal to this value c, and therefore greater or equal to $G(x,y,z)$. Thus, we have shown that the inequality (1), i.e., $A(x,y,z)\geq G(x,y,z)$, holds whenever x,y,z are such that $G(x,y,z)=c$. Since c was arbitrary, we have proven that this inequality holds for all (nonnegative) x,y,z.

Remarks. The argument showing that the extremum obtained by the Lagrange multiplier method is indeed a minimum (i.e., Step II above) is a key part of the solution to this problem. It is intellectually the most challenging aspect of the problem and was worth half the credit. As indicated in the statement of the problem, this argument had be ironclad, rocksolid, and bulletproof. A guess (no matter how educated) doesn't cut it, nor does simply saying that it is a minimum make it so.

In fact, whether the value found in Step I is a minimum or a maximum makes all the difference in the world. If it had been a maximum, the conclusion would be that $A(x,y,z)\leq G(x,y,z)$, so the G-formula would always yield the larger overall score, which is the exact opposite of the above (correct) answer.

Alternative approach: In the above approach the constraint was $G(x,y,z)=c$ and $A(x,y,z)$ was the function to be optimized. Alternatively, one can try to optimize the function $G(x,y,z)$ subject to the constraint $A(x,y,z)=c$. This leads to a very similar system of Lagrange equations as in Step I above, with the same solution: $(x,y,z)=(c,c,c)$ is the extremal point, and c the extremal value. However, as before, this is not the end of the story, and an additional argument is needed to determine whether the point found is a maximum or a minimum. (In this case, in order to obtain the inequality (1), one needs to show that this value is a *maximum*, not a minimum.)