Problem 1

Let $\vec{F}$ be the vector field defined by $\vec{F} = r^2\vec{r}$ (in 3-dimensional space).

(i) Find the **divergence** of $\vec{F}$.

Solution: We have

$$\begin{aligned}
\vec{F} &= (x^2 + y^2 + r^2)\langle x, y, z\rangle \\
&= \langle x^3 + y^2x + z^2x, x^2y + y^3 + z^2y, x^2z + y^2z + z^3\rangle = \langle P, Q, R\rangle, \\
\nabla \cdot F &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\
&= (3x^2 + y^2 + z^2) + (3y^2 + x^2 + z^2) + (3z^2 + x^2 + y^2) \\
&= 5(x^2 + y^2 + z^2) = \boxed{5r^2}
\end{aligned}$$

(ii) Find the **curl** of $\vec{F}$.

Solution: Using the above formula for $\vec{F}$, we get

$$\begin{aligned}
\nabla \times \vec{F} &= \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \langle 2yz - 2zy, -2xz + 2zx, 2xy - 2yx\rangle = \boxed{\langle 0, 0, 0\rangle}
\end{aligned}$$

Problem 2

(For this problem just state the desired formulas in the requested form. No proofs or justification required.)

(i) Let R be a finite simply connected 2-dimensional region, and C its boundary curve, in counterclockwise orientation. Express the area of R as a line integral over C. Your answer should be in the form $\oint_C * \, d*$, with appropriate expressions in place of the asterisks.

Solution: By the Green's Theorem Area Formula, $A(R)$ is given by any of the following expressions:

$$\oint_C x dy, \quad \oint_C (-y) dx, \quad \frac{1}{2} \oint_C x dy - y dx.$$

(Any one of these forms is acceptable as answer.)

(ii) Let $\vec{F}$ be a 3-dimensional vector field, and let S be a surface parametrized by a function $\vec{r}(u, v)$ for (u, v) in some region R in the uv-plane. Express the (outward) flux of $\vec{F}$ through S as a double integral with respect to u and v. Your answer should be in the form $\iint\limits_R * \, du \, dv$, with an appropriate expression in place of the asterisk.

Solution: The surface area element is $dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv$, so the flux of $\vec{F}$ through S is

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot \vec{n} |\vec{r}_u \times \vec{r}_v| \, du \, dv.$$

Alternatively, the flux can be expressed as

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv,$$

(Either of the above formulas is acceptable. The second follows from the first since the vector $\vec{r}_u \times \vec{r}_v$ is normal to the surface, and so $\vec{n} = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$.)

Problem 3

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$, where $\vec{F} = \langle y^2, x \rangle$ and C is the straight line segment from $(1, 0)$ to $(0, 2)$.

Solution: The line segment can be parametrized by

$$x(t) = 1 - t, \quad y(t) = 2t, \quad 0 \leq t \leq 1.$$

With this parametrization we get

$$\begin{aligned}\int_C \vec{F} \cdot \vec{T}\, ds = \int_C y^2 dx + x dy &= \int_0^1 (2t)^2(-1)dt + (1-t)(2)dt \\ &= \int_0^1 (-4t^2 - 2t + 2)dt = -\frac{4}{3} - \frac{2}{2} + 2 = \boxed{-\frac{1}{3}}\end{aligned}$$

Problem 4

Find the **center of mass** of a wire in the xy-plane shaped like the semicircle $x^2 + y^2 = a^2$, $y \geq 0$, with mass density $\delta(x, y) = y$.

Solution: We parametrize the semicircle by

$$x(t) = a\cos t, \quad y(t) = a\sin t, \quad 0 \leq t \leq \pi.$$

Then

$$ds = \sqrt{x'(t)^2 + y'(t)^2}dt = \sqrt{(-a\sin t)^2 + (a\cos t)^2}dt,$$

and

$$\begin{aligned} m &= \int_C \delta ds = \int_0^\pi (a\sin t)a\,dt = -a^2\left[\cos t\right]_0^\pi = \boxed{2a^2}, \\ \overline{y} &= \frac{1}{m}\int_C y\delta ds = \frac{1}{m}\int_0^\pi (a\sin t)(a\sin t)a\,dt \\ &= -\frac{a^3}{m}\int_0^\pi \frac{1}{2}\left(\cos 2t - 1\right)dt \\ &= -\frac{a^3}{2m}\left[\frac{\sin 2t}{2} - t\right]_0^\pi = \frac{a^3\pi}{2m} = \boxed{\frac{\pi}{4}a} \end{aligned}$$

By symmetry, $\overline{x} = 0$, so the center of mass is $\boxed{(0, \frac{\pi}{4}a)}$. (Sanity check: Since $\pi/4$ is around 0.78, the point found falls inside the semicircle and its "height", $\overline{y}$, is about 78 percent of the total height of the semicircle. Since there is more mass in the top half of the semicircle, this is a perfectly plausible result.)

Problem 5

Evaluate the integral

$$\oint_C (y + \sin x)dx + (x^2 + x + \cos y)dy,$$

where C is the boundary of the region in the unit square enclosed by the curves $y = x^2$ and $x = y^2$, oriented in counterclockwise direction.

Solution: Let R denote the given region. By Green's theorem, the given integral is

$$\begin{aligned}\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_R ((2x+1) - 1)dA = \iint_R 2x dA \\ &= \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} 2x\, dy\, dx = \int_{x=0}^{x=1} 2x \left(x^{1/2} - x^2\right) dx \\ &= \int_0^1 (2x^{3/2} - 2x^3)dx = 2 \cdot \frac{1}{5/2} - 2 \cdot \frac{1}{4} = \boxed{\frac{3}{10}}\end{aligned}$$

Problem 6

Let

$$\vec{F} = \langle x^3 + xy^2 + x, y^3 + yx^2 + y \rangle.$$

Show that $\vec{F}$ is conservative and find a potential for $\vec{F}$.

Solution: Let P and Q denote the x- and y-components of $\vec{F}$. We have

$$\frac{\partial Q}{\partial x} = 2xy, \quad \frac{\partial P}{\partial y} = 2xy,$$

so $\partial Q/\partial x = \partial P/\partial y$, and $\vec{F}$ is therefore conservative. To find a potential for $\vec{F}$, we use the partial integration method to find a function $f(x, y)$ such that $f_x = P$ and $f_y = Q$. We first integrate P with respect to x to get

$$\begin{aligned} f(x,y) &= \int f_x dx = \int P dx \\ &= \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2}{2} + g(y), \end{aligned}$$

where $g(y)$ is a function of y. Next, we differentiate the above expression for $f(x, y)$ with respect to y and set the result equal to Q:

$$f_y(x, y) = x^2y + g'(y) = Q = y^3 + yx^2 + y.$$

Solving for $g'(y)$, we get

$$\begin{aligned} g'(y) &= y^3 + y, \\ g(y) &= \frac{y^4}{4} + \frac{y^2}{2} + C, \\ f(x,y) &= \frac{1}{4}(x^4 + y^4) + \frac{1}{2}x^2y^2 + \frac{1}{2}(x^2 + y^2) + C \\ &= \frac{1}{4}(x^2 + y^2)^2 + \frac{1}{2}(x^2 + y^2) + C = \boxed{\frac{1}{4}r^4 + \frac{1}{2}r^2 + C}, \end{aligned}$$

where C is an arbitrary constant.

Problem 7

Express the flux of the vector field

$$\vec{F} = \langle 2x, 4y, 1z \rangle$$

through the triangular surface with vertices at $(2, 0, 0)$, $(0, 4, 0)$, $(0, 0, 1)$ as a double integral of the form

$$\int_{x=*}^{x=*} \int_{y=*}^{y=*} * \, dy \, dx,$$

with *concrete* expressions in the place of the asterisks. (The expressions should be explicit, i.e., numerical constants or functions of x and y, not generic formulas involving $\vec{F}$, $\vec{T}$, $\vec{n}$, etc. The point of the problem is to reduce the task of finding the flux to a routine (though possibly rather tedious) computation of a double integral in x and y, without having you waste time by actually carrying out this routine computation. The intellectual content of the problem is contained in this reducation, and you should carry it out carefully; use the back of the page if you need additional space.)

Solution: The flux is given by the integral $\iint_S \vec{F} \cdot \vec{n} \, dS$. To convert this integral to an integral of the desired form we first need to compute dS, $\vec{n}$, $\vec{F} \cdot \vec{n}$.

To compute dS, note that the given triangle lies in a plane of the form $ax + by + cz = 1$. The coefficients a, b, c, here can be obtained by plugging in the given vertices. The first vertex, $(2, 0, 0)$, gives $a \cdot 2 = 1$, so $a = 1/2$. Similarly we get $b = 1/4$, and $c = 1$, so the equation of the plane is $(*)$ $\frac{1}{2}x + \frac{1}{4}y + z = 1$ or $(**)$ $z = f(x, y) = 1 - \frac{1}{2}x - \frac{1}{4}y$. From $(**)$ we get the surface area element,

$$dS = \sqrt{1 + f_x^2 + f_y^2} dA = \sqrt{1 + (1/2)^2 + (1/4)^2} dA = \frac{\sqrt{21}}{4} dA.$$

From $(*)$ we can read off a normal vector,

$$\vec{n} = \frac{\langle 1/2, 1/4, 1 \rangle}{\sqrt{1/4 + 1/16 + 1}} = \frac{1}{\sqrt{21}} \langle 2, 1, 4 \rangle.$$

Therefore,

$$\vec{F} \cdot \vec{n} = \frac{1}{\sqrt{21}} (4x + 4y + 4z),$$

Substituting the given equation $z = 1 - \frac{1}{2}x - \frac{1}{4}y$, we get that *on the surface*

$$\vec{F} \cdot \vec{n} = \frac{4}{\sqrt{21}} (x + y + 1 - \frac{1}{2}x - \frac{1}{4}y) = \frac{4}{\sqrt{21}} (1 + \frac{1}{2}x + \frac{3}{4}y).$$

Hence, the desired flux integral

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, dS &= \iint_R \frac{4}{\sqrt{21}} \left(1 + \frac{1}{2}x + \frac{3}{4}y \right) \frac{\sqrt{21}}{4} \, dA \\ &= \iint_R \left(1 + \frac{1}{2}x + \frac{3}{4}y \right) dA, \end{aligned}$$

where R is the region in the xy-plane over which the given triangular surface S lies. Now, R is a triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 4)$, described by the inequalities

$$R : \quad \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 4 - 2x \end{cases}$$

Thus, the double integral $\iint_R$ can be written as an iterated integral $\int_{x=0}^{2} \int_{y=0}^{4-2x}$, and we get

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \int_{x=0}^{2} \int_{y=0}^{4-2x} \left(1 + \frac{1}{2}x + \frac{3}{4}y \right) dy \, dx.$$

which is an integral of the requested form.