1. *(10) Quickies (2 points each). Describe the regions that are given by the following equations. For each case circle one of the five given choices.*
(a) $\phi = 1$ in spherical coordinates:
a cylinder; a half-plane; a plane; a cone; a sphere; none of the above

A cone

(b) $\rho = 1$ in spherical coordinates:
a cylinder; a half-plane; a plane; a cone; a sphere; none of the above

A sphere

(c) $z^2 + r^2 = 1$ in cylindrical coordinates:
a cylinder; a half-plane; a plane; a cone; a sphere; none of the above

A sphere ($x^2 + y^2 + z^2 = 1$)

(d) $z = r^2$ in cylindrical coordinates:
a cylinder; a half-plane; a plane; a cone; a sphere; none of the above

None of the above (a paraboloid)

(e) $r = 1$ in cylindrical coordinates:
a cylinder; a half-plane; a plane; a cone; a sphere; none of the above

A cylinder

[Parts (c) and (d) are Problems 43 and 37 of 12.7. The other parts are variations of Problems 31 - 36 of 12.7]

2. *(10) The following two subproblems are independent of each other.*
(i) *(5) Convert the equation $\rho = \cos\phi$ to an equation in rectangular coordinates, simplifying as much as possible, and identify the surface represented by this equation. (Be specific, saying for example, "a cylinder of radius 2, about the z-axis", instead of "a cylinder".)* [Except for a factor 2 in front of $\cos\phi$ this is the sphere occurring in connection with the "ice cream cone" problem, Example 4 of 15.9; it also occurs as Problem 42 in 12.7.]

Multiplying the given equation by ρ gives $\rho^2 = \rho\cos\phi$, which is equivalent to the equation $x^2 + y^2 + z^2 = z$, or $\boxed{x^2 + y^2 + (z - 1/2)^2 = 1/4}$. The latter is the equation of a $\boxed{\text{sphere of radius } 1/2, \text{ with center } (0, 0, 1/2)}$.

(ii) *(5) Express a cone centered at the origin, opening in the direction of the positive z-axis, and forming an angle of $\pi/3$ with the z-axis, in (a) spherical coordinates, and (b) cylindrical coordinates, simplifying as much as possible.*

In spherical coordinates, the cone is given by $\boxed{\phi = \pi/3}$; in cylindrical coordinates, it is given by $\boxed{z = r/\sqrt{3}}$ (since $r/z = \tan(\pi/3) = (\sqrt{3}/2)/(1/2) = \sqrt{3}$).

3. *(10) In each of the following subproblems set up integrals of the form* $\int_*^* \int_*^* * dydx$ *(in this order, with the five asterisks replaced by appropriate numbers or expressions) for the quantity described in the problem. Do not evaluate these integrals.*

(i) *(5) The volume* V *under the surface* $z = xy$ *and above the triangle with vertices* $(1,1)$, $(4,1)$, *and* $(1,2)$ [Problem 21 of 15.3, an assigned HW problem]

Drawing a picture, one sees that the region D represented by this triangle lies between the vertical lines $x = 1$ and $x = 4$, is bounded below by the line $y = 1$ and from above by the line $y = -x/3 + 7/3$. Hence D is determined by the inequalities $1 \leq x \leq 4$, $1 \leq y \leq (7-x)/3$, and the volume in question is the integral of $f(x,y) = xy$ over D, i.e., $\boxed{V = \iint_D xy dA = \int_1^4 \int_1^{7/3-x/3} xy dy dx}$.

(ii) *(5) The mass* m *of a lamina occupying the triangle with vertices* $(0,0)$, $(0,3)$, *and* $(1,1)$, *if the mass density at any point inside the triangle is equal to the distance of that point from the* x*-axis.*

The distance of a point (x,y) to the x-axis is y, so the mass density function is $\rho(x,y) = y$. The given triangle is described by the inequalities $0 \leq x \leq 1, x \leq y \leq 3-2x$. (Draw picture!) Hence the mass of the lamina is $\boxed{m = \iint_D \rho(x,y) dA = \int_0^1 \int_x^{3-2x} y dy dx}$.

4. *(10) Evaluate the integral* $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} dydx$.
[Problem 27 in 15.4]

The domain of integration is the portion of the unit disk that lies in the first quadrant. In polar coordinates, this "quarter disk" is given by the inequalities $0 \leq r \leq 1, 0 \leq \theta \leq \pi/2$, and the integral becomes $\int_0^{\pi/2} \int_0^1 e^{r^2} r dr d\theta$. Substituting $u = r^2, du = 2rdr$ in the inner integral, we get $\int_0^{\pi/2} \int_0^1 e^u (1/2) du d\theta = \int_0^{\pi/2} \frac{1}{2}(e-1) d\theta = \boxed{\frac{\pi(e-1)}{4}}$

5. *(10) Find the volume of the region that lies inside the cylinder* $x^2 + y^2 = 1$*, below the plane* $z = 4$*, and above the surface* $z = 1 - x^2 - y^2$*.*
[Variation of Example 1 in 15.8]

It is easiest to compute the volume via double integrals. The given region is the region lying over the unit disk $D : x^2 + y^2 \leq 1$ in the xy-plane, and between the graphs of the functions $f(x, y) = 4$ and $g(x, y) = 1 - x^2 - y^2$. Its volume is therefore given by the formula $V = \iint_D (f(x, y) - g(x, y))dA = \iint_D (4 - (1 - x^2 - y^2))dA$. Converting to polar coordinates, the integral becomes $\int_0^{2\pi} \int_0^1 (3 + r^2) r dr d\theta$. The inner integral (involving r) is $\int_0^1 (3r + r^3)dr = (3/2) + (1/4) = 7/4$, and integrating over θ gives a factor 2π. Hence $\boxed{V = \frac{7}{2}\pi}$. [Alternatively, one could compute the volume as a triple integral $V = \iiint 1 dV$. Using cylindrical coordinates, the integral becomes $\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 dz r dr d\theta = \int_0^{2\pi} \int_0^1 (3 + r^2) r dr d\theta$, which is the same as the above double integral in r and θ.]

6. *(10) Find the* **area** *of the surface of the part of the sphere* $x^2 + y^2 + z^2 = 25$ *that lies above the plane* $z = 4$*. (Note that this problem asks for a surface area, not a volume.)*
[A problem on one of the practice exams (Exam 3/99).]

The surface in question S lies above a disk D centered at the origin. To find the radius R of that disk, use a bit of elementary trigonometry: The radius R and the height $z = 4$ form the two legs of a right triangle whose hypotenuse has length equal to the radius of the sphere, i.e., 5, so $R^2 + 4^2 = 5^2$, or $R = \sqrt{5^2 - 4^2} = 3$. The surface is the part of the surface $z = \sqrt{25 - x^2 - y^2} = f(x, y)$ that lies above this disk D. A simple calculation shows $\sqrt{f_x^2 + f_y^2 + 1} = 5/\sqrt{25 - x^2 - y^2} = 5/\sqrt{25 - r^2}$, so using the formula for the area of a surface $z = f(x, y)$ above a region D and switching to polar coordinates, we get $A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} dA = \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r dr d\theta = 5 \int_0^{2\pi} (-(25 - r^2)^{1/2}) \Big|_0^3 d\theta = 5 \cdot (2\pi)(\sqrt{5^2} - \sqrt{5^2 - 3^2}) = \boxed{10\pi}$. (Note that this problem required using the formula for the area of a surface from 15.6, which involves a **double** integral; a triple integral would not be appropriate here.)

7. *(10) A cylindrical drill with radius 3 is used to bore a hole through the center of a sphere of radius 5. Find the volume of the remaining ring-shaped solid (a sphere with a hole). (Hint: position the sphere so that its center is at the origin and the axis of the cylindrical hole is the z-axis.)*
[A simplified version of 26(a) of 15.4 (worked out in the discussion sections)]

Let R denote the upper half of the solid in question, as described in the hint. Then R is the region below the surface $z = \sqrt{25 - x^2 - y^2}$ (the surface of a hemisphere of radius 5 centered at the origin) and above the region D in the xy-plane given by the inequality $9 \le x^2 + y^2 \le 25$ (the part of the "base" of the hemisphere that remains after drilling a hole of radius 3). Hence $V(R) = \iint_D \sqrt{25 - x^2 - y^2} dA$. In polar coordinates, D is given by $3 \le r \le 5, 0 \le \theta \le 2\pi$ and $\sqrt{25 - x^2 - y^2} = \sqrt{25 - r^2}$, so

$$V(R) = \int_0^{2\pi} \int_3^5 \sqrt{25 - r^2}\, r dr d\theta = \int_0^{2\pi} \int_{16}^0 \sqrt{u}(-1/2) du d\theta$$
$$= \frac{1}{2} \int_0^{2\pi} \int_0^{16} u^{1/2} du d\theta = \frac{1}{2} \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_{y=0}^{16} d\theta = \frac{1}{3} \int_0^{2\pi} (16)^{3/2} d\theta = \frac{128\pi}{3},$$

The volume of the complete "sphere with hole" is twice $V(R)$, i.e.,
$\boxed{V = \frac{256}{3}\pi}$.

Remark: One cannot compute this volume by subtracting the volume of a cylinder from that of a sphere, since the "hole" is not an ordinary cylinder with a flat roof, but one with dome-shaped roof. One could, of course, fix this by computing the volume of such a cylinder with a dome-shaped roof, but that calculation would amount to the same as the one carried out above, the only difference being that the limits in the r-integral above are 0 and 3 instead of 3 and 5.