Math 241 Section F1H, Fall 2007

Functions, limits, and continuity on n-dimensional spaces

n-dimensional spaces:

- **n-dimensional Euclidean space, $\mathbf{R}^n$:** The n-dimensional (Euclidean) space, $\mathbf{R}^n$, is defined as the set of all n-tuples $\mathbf{x} = (x_1, x_2, \dots, x_n)$ of real numbers. Here n, the **dimension** of the space, can be any positive integer.
- **Euclidean norm, $|\mathbf{x}|$:** Given an element $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in $\mathbf{R}^n$, its Euclidean norm (or magnitude/length) is defined as

$$\boxed{|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$$

- **Functions from $\mathbf{R}^n$ to $\mathbf{R}^m$:** A **function f from $\mathbf{R}^n$ to $\mathbf{R}^m$**, denoted by $f : \mathbf{R}^n \to \mathbf{R}^m$, is a rule that assigns to every element $\mathbf{x}$ in $\mathbf{R}^n$ a unique element $f(\mathbf{x})$ in $\mathbf{R}^m$. In other words, f is a "black box" that takes as input an element $\mathbf{x}$ in $\mathbf{R}^n$ and produces as output an element $f(\mathbf{x})$ in $\mathbf{R}^m$. More generally, one can consider functions $f : D \to \mathbf{R}^m$, defined on some subset D of $\mathbf{R}^n$, the **domain** of the function.

Limits:

- **$\epsilon - \delta$ definition:** Given a function $f : \mathbf{R}^n \to \mathbf{R}^m$ and a point $\mathbf{a}$ in $\mathbf{R}^n$, we say that **the limit of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is equal to $\mathbf{b}$**, and we write $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}$, if the following holds (see the picture below):

For every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that the following holds:
If $0 < |\mathbf{x} - \mathbf{a}| < \delta$, then $|f(\mathbf{x}) - \mathbf{b}| < \epsilon$.

The bars, $|\dots|$, here denote the Euclidean norm, as defined above. Note that the left inequality, $0 < |\mathbf{x} - \mathbf{a}|$, explicitly excludes the case when $\mathbf{x}$ is *exactly* equal to $\mathbf{a}$ from any consideration. *Thus, the value of $f(\mathbf{x})$ at $\mathbf{x} = \mathbf{a}$ has no bearing whatsoever on the limit $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$; in fact, $f(\mathbf{x})$ need not even be defined when $\mathbf{x} = \mathbf{a}$.*

- **See also:** Edwards/Penney, Section 12.3 (for the case of functions of two variables, $f : \mathbf{R^2} \to \mathbf{R^1}$), and Appendix A-19 (for an extended discussion of the case of one variable functions, $f : \mathbf{R^1} \to \mathbf{R^1}$).

Continuity:

- **Formal definition:** A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is said to be **continuous at the point $\mathbf{a}$** if

$$\boxed{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})}$$

i.e., if *all* of the following hold:

(i) The limit of $f(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ exists, and is equal to some value $\mathbf{b}$, say.
(ii) $f(\mathbf{a})$ exists, i.e., the function $f(\mathbf{x})$ is defined at the point $\mathbf{x} = \mathbf{a}$.
(iii) $f(\mathbf{a})$ is equal to $\mathbf{b}$.

f is said to be **continuous in a domain D in $\mathbf{R}^n$** if it is continuous at every point $\mathbf{a}$ in D.