Problem 1 (12 points)

(i) Express the point $(x, y, z) = (1, 1, -\sqrt{2})$ in **spherical coordinates** (ρ, ϕ, θ).

Solution: First, we have $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 1^2 + (-\sqrt{2})^2} = \sqrt{4} = 2$. Next, using $z = \rho\cos\phi$, with $z = -\sqrt{2}$ and $\rho = 2$, we get $\cos\phi = -1/\sqrt{2}$, which implies $\phi = 3\pi/4$ since ϕ has to be in the interval $[0, \pi]$. Finally, since $x = y = 1$, the x- and y-coordinates of the point are on the diagonal in the first quadrant, we have $\theta = \pi/4$. Thus, $\boxed{(\rho, \phi, \theta) = (2, 3\pi/4, \pi/4)}$.

(ii) Express the **half-plane** $y = -x$, where $x \geq 0$, as an equation in **cylindrical coordinates**, **simplifying as much as possible**.

Solution: The equation $y = -x$ with $x \geq 0$ represents the diagonal in the **fourth quadrant** of the xy-plane, so the given half-plane is a vertical plane that intersects the xy-plane at this diagonal. In cylindrical coordinates, this plane is given by $\boxed{\theta = 7\pi/4}$.

Remarks. The above geometric argument is the easiest way to get the equation. Alternatively, substituting $x = r\cos\theta$, $y = r\sin\theta$ into the given equation, one gets $\tan\theta = -1$. Solving this equation for θ gives two solutions, $\theta = 3\pi/4$ or $\theta = 7\pi/4$. However, with the first of these, $\theta = 3\pi/4$, the number $x = r\cos\theta$ would be negative, contradicting the condition $x \geq 0$. Thus, we are left with the second solution, $\theta = 7\pi/4$, which is the one obtained above.

(iii) Express the surface given by the equation (in spherical coordinates) $\rho = \cos\phi$ as an equation in **cylindrical coordinates**, **simplifying as much as possible**.

Solution: Multiplying by ρ and using $\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$ and $\rho\cos\phi = z$ gives $r^2 + z^2 = z$; or (after completing the square) $\boxed{r^2 + (z - 1/2)^2 = 1/4}$, or $\boxed{r = \sqrt{1/4 - (z - 1/2)^2}}$ (not $r = \pm\sqrt{\ldots}$ since r is positive).

Problem 2 (8 points)

(i) (Multiple choice) Circle the correct formula for $\overline{x}$, the **x-coordinate of the center of mass** of a lamina occupying a region R in the xy-plane and having mass density $\delta = \delta(x, y)$.

(a) $\overline{x} = \dfrac{\iint_R x\delta dA}{\iint_R \delta dA}$ (b) $\overline{x} = \dfrac{\iint_R x\delta dA}{\iint_R x dA}$ (c) $\overline{x} = \dfrac{\iint_R x\delta dA}{\iint_R 1 dA}$

(d) $\overline{x} = \iint_R x\delta dA$ (e) $\overline{x} = \iint_R x^2\delta dA$ (f) $\overline{x} = \iint_R y^2\delta dA$

Solution: (a) $\boxed{\overline{x} = \frac{1}{m}\iint\limits_R x\delta dA = \frac{\iint\limits_R x\delta dA}{\iint\limits_R \delta dA}}$

Grading note: This part was worth 4 points. Full credit (4 points) was given for the correct answer (a), 1 point for the answers (b), (c), or (d) (all incorrect, but close enough to the correct formula to warrant some partial credit), and no credit for any of the other answers.

(ii) (Multiple choice) Circle the correct formula for the **moment of inertia about the x-axis**, I_x, of a lamina occupying a region R in the xy-plane and having mass density $\delta = \delta(x, y)$.

(a) $I_x = \frac{1}{m}\iint\limits_R x\delta dA$ (b) $I_x = \frac{1}{m}\iint\limits_R x^2\delta dA$ (c) $I_x = \frac{1}{m}\iint\limits_R y^2\delta dA$

(d) $I_x = \iint\limits_R x\delta dA$ (e) $I_x = \iint\limits_R x^2\delta dA$ (f) $I_x = \iint\limits_R y^2\delta dA$

Solution: (f) $\boxed{I_x = \iint_R y^2\delta dA}$

Remarks. Note that the formulas for the moments of inertia do not involve a factor $1/m$, in contrast to the formulas for the center of mass. Also, the integrand in any type moment of inertia is the **square of the distance to the axis of rotation**. In this case, the axis of rotation is the x-axis, and the distance of a general point (x, y) to the x-axis is its y-coordinate, so the correct factor inside the integral is y^2.

Grading note: This part was worth 4 points. Full credit (4 points) was given for the correct answer (f), 1 point for answers (c) or (e) (all incorrect, but close enough to the correct formula to warrant some partial credit), and no credit for any of the other answers.

Problem 3 (10 points)

Evaluate the integral $\int_{y=0}^{1}\int_{x=3y}^{3} e^{x^2}dxdy$ by reversing the order of integration.

Solution: First read off the integration limits in the given form of integral: $\boxed{0 \le y \le 1, 3y \le x \le 3}$. Next, sketch the region R determined by these inequalities; it is a triangle with vertices $(0, 0)$, $(3, 0)$, and $(3, 1)$ (see below). To obtain an integral in reverse order, with the x-integration outside, "sweep out" this region vertically, and read off the limits on x and y corresponding to this sweep: $\boxed{0 \le x \le 3, 0 \le y \le x/3}$. Using these limits, rewrite the integral with the x-integral on the outside, and the y-integral inside, and evaluate it:

$$\begin{aligned}\int_{y=0}^{1}\int_{x=3y}^{3} e^{x^2}dxdy &= \int_{x=0}^{3}\int_{y=0}^{x/3} e^{x^2}dydx \\ &= \int_{x=0}^{3}\left[e^{x^2}y\right]_{y=0}^{y=x/3}dx \\ &= \int_{0}^{3}\left(\frac{x}{3}\right)e^{x^2}dx \\ &= \int_{u=0}^{u=9}\frac{1}{6}e^{u}du \qquad (\text{set } u=x^2,\ du=2xdx) \\ &= \boxed{\frac{1}{6}(e^9-1)}.\end{aligned}$$

Remark. Note that, in the given form, the integral cannot be computed, since the function e^{x^2} has no (elementary) anti-derivative. (There is, in fact, a theorem to that effect; no matter what integration tricks you try, the theorem guarantees that will not succeed in integrating e^{x^2}.)

Problem 4 (10 points)

Let E denote the (3-dimensional) region between the two cylinders $x^2+y^2=1$ and $x^2+y^2=4$ that is bounded from above by the surface $z=x^2$ and from below by the xy-plane. Find the volume of E.

Solution: The volume is given by $V(E)=\iint_R(z_{TOP}-z_{BOT})dA$, where R is the (ring-shaped) region in the xy-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$, $z_{TOP}=x^2$ and $z_{BOT}=0$ are the functions representing the "top" and "bottom" of the region E. The shape of R suggests to use polar coordinates. In polar coordinates, R is given by the inequalities $0\le\theta\le 2\pi, 1\le r\le 2$, and the integrand is $z_{TOP}-z_{BOT}=x^2-0=(r\cos\theta)^2$. Thus,

$$\begin{aligned}V=\iint_R x^2dA &= \int_{\theta=0}^{2\pi}\int_{r=1}^{2}(r\cos\theta)^2 rdrd\theta \\ &= \int_0^{2\pi}(\cos\theta)^2\left[\left(\frac{1}{4}r^4\right)\right]_{r=1}^{2}d\theta = \int_0^{2\pi}(\cos\theta)^2\frac{2^4-1^4}{4}d\theta \\ &= \frac{15}{4}\int_0^{2\pi}(\cos\theta)^2d\theta = \frac{15}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos(2\theta))d\theta \\ &= \frac{15}{8}\left[\theta+\frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} \\ &= \boxed{\frac{15\pi}{4}=3.75\pi}\end{aligned}$$

Remark: Alternatively, one can use a triple integral and cylindrical coordinates to compute the volume:

$$\begin{aligned} V(E) &= \iiint_E 1dV = \int_{\theta=0}^{2\pi}\int_{r=1}^{2}\int_{z=0}^{(r\cos\theta)^2} 1\, rdzdrd\theta \\ &= \int_{\theta=0}^{2\pi}\int_{r=1}^{2} [zr]_{z=0}^{z=(r\cos\theta)^2} 1\, drd\theta \\ &= \int_{\theta=0}^{2\pi}\int_{r=1}^{2} (r\cos\theta)^2 rdrd\theta, \end{aligned}$$

The latter integral is the same as that in the above double-integral approach.

Problem 5 (10 points)

Find the mass of a solid sphere of radius 1 centered at the origin whose mass density is given by the function

$$\delta(x,y,z) = e^{-(x^2+y^2+z^2)^{3/2}}.$$

Solution: In spherical coordinates, the density is simply $\delta = e^{-\rho^3}$, and the sphere is described by the inequalities $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, $0 \le \rho \le 1$. Thus

$$\begin{aligned} m &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} e^{-\rho^3}\rho^2 \sin\phi d\rho d\phi d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi \int_{u=0}^{1} e^{-u}(1/3)dud\phi d\theta \quad (\text{set } u=\rho^3,\ du = 3\rho^2 d\rho) \\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi(1/3)(1-e^{-1})d\phi d\theta \\ &= \int_{0}^{2\pi}\int_{0}^{\pi} [-\cos\phi]_{\phi=0}^{\pi}(1/3)(1-e^{-1})d\theta \\ &= (2\pi)(2)(1/3)(1-e^{-1}) = \boxed{\frac{4\pi}{3}(1-e^{-1})} \end{aligned}$$

Remarks. If this looked vaguely familiar, it's no accident: the above triple integral is the same as that of Problem 7 in 13.7, a problem covered in the hw assignments and discussion sections.

Note that this problem **cannot be done with double integrals**. Only volumes can be computed (in certain cases) with double integrals, but the given problem asks for the mass of a three-dimensional object with a mass density function $\delta(x,y,z)$. The triple-integral approach is the only option here.

The problem **cannot be done with cylindrical or rectangular coordinates**. Using spherical coordinates is the only way to evaluate the integral. Because of the shape of the object (a sphere), spherical coordinates are in fact the obvious, and most natural, choice. If one tries to use cylindrical coordinates one ends up with integrals that cannot be evaluated.

Problem 6 (10 points)

In the following subproblems T denotes the solid consisting of all points (x,y,z) satisfying $x \ge 0, y \ge 0, z \ge 0$ and $x^2+y^2+z^2 \le 1$.

(i) Express (but do not evaluate) the volume of T as a **double** integral in **rectangular** coordinates, i.e., in the form $\int_*^* \int_*^* * \, dxdy$ (in this order), with appropriate expressions in place of the 5 asterisks.

Solution: The region T is the first octant portion of the standard unit sphere. It "sits" over the quarter disk $0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$, and its "roof" is given by $z = \sqrt{1-x^2-y^2}$. By the double-integral formula for a volume, $V(T) = \iint_R f(x,y)$, where R is the above quarter disk and $f(x,y) = \sqrt{1-x^2-y^2}$. Thus,

$$V(T) = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} \, dydx.$$

(ii) Express (but do not evaluate) the volume of the region T as an integral in **cyclindrical** coordinates of the form $\int_*^* \int_*^* \int_*^* * \, drdzd\theta$ (in this order), with appropriate expressions in place of the 7 asterisks.

Solution: In cylindrical coordinates, the inequalities $x \ge 0, y \ge 0$ reduce the θ-range to $0 \le \theta \le \pi/2$, and the inequalities $z \ge 0$ and $x^2+y^2+z^2 \le 1$ translate to $z \ge 0$ and $r^2 + z^2 \le 1$, or $0 \le z \le 1, 0 \le r \le \sqrt{1-z^2}$. (Note that r is always positive, so the lower limit for r is 0.) Thus, using the triple-integral formula for a volume, $V(T) = \iiint_T 1 dV = \iiint_T r drdzd\theta$, we get

$$V(T) = \int_{\theta=0}^{\pi/2} \int_{z=0}^{1} \int_{r=0}^{\sqrt{1-z^2}} r drdzd\theta$$